Calculus Review: Derivatives and Integrals

PSTAT 120A: Summer 2022

Ethan P. Marzban

University of California, Santa Barbara

Derivatives

Derivatives

- Recall that the derivative of a function $f(x)$ at the point a, notated $f'(a)$, can be interpreted as the "slope" of the function at the point a.
- More properly, it is the slope of the line tangent to $f(x)$ at the point $(a, f(a))$.

Definition: Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example

- As an example, let's compute the derivative of $f(x) = x^2$ from the definition: we first write

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h\end{aligned}$$

- Therefore

$$f'(x) = \lim_{h\to 0}(2x+h) = \boxed{2x}$$

Common Derivatives

Theorem: Useful Derivatives

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln(a)$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \dfrac{1}{x}$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x)$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x)$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x} \arctan(x) = \dfrac{1}{1+x^2}$

Derivative Rules

Theorem: Multiplication Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Theorem: Division Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Theorem: Chain Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f\big(g(x)\big)\right] = f'(g(x)) \cdot g'(x)$$

Example

- As an example, suppose we wish to evaluate $\frac{d}{dx}\left[xe^{\cos(x)}\right]$
- By the Product Rule,

$$\frac{d}{dx}\left[xe^{\cos(x)}\right] = (1)e^{\cos(x)} + x \cdot \frac{d}{dx}\left[e^{\cos(x)}\right]$$

- To evaluate the final derivative on the RHS, we use the Chain Rule:

$$\frac{d}{dx}\left[xe^{\cos(x)}\right] = (1)e^{\cos(x)} + x \cdot e^{\cos(x)} \cdot (-\sin(x))$$

or, cleaning up terms a bit,

$$\frac{d}{dx}\left[xe^{\cos(x)}\right] = e^{\cos(x)}\left[1 - x\sin(x)\right]$$

L'Hôpital's Rule

Formula: L'Hôpital's Rule

$$\lim_{x \to c} \left[\frac{f(x)}{g(x)} \right] = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided we have an indeterminate form of the type $0/0$ or ∞/∞.

Integrals

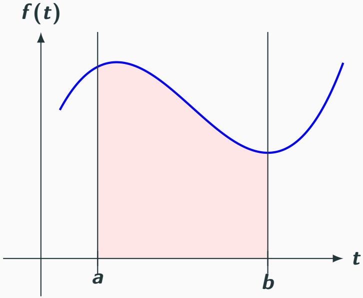
Indefinite Integrals

- What is the "opposite" of differentiation? In other words, given a function $f(x)$ can I find a function $F(x)$ such that $F'(x) = f(x)$?
- Yes! Such a function is called a **antiderivative**.
- For example, if $f(x) = 2x$, I can recognize that $\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$, meaning $F(x) = x^2$ is an antiderivative of $f(x) = 2x$.
- Wait a minute; an antiderivative? Are antiderivatives not unique?
- No, they are not. Take our $f(x) = 2x$ example again. It is true that $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + 4) = 2x$. Therefore, by our definition of an antiderivative, $F(x) = x^2 + 2$ is also an antiderivative.
- In general, a function $f(x)$ has a family of antiderivatives, differing by a constant. So, for our $f(x) = 2x$ example, we would say that the class of antiderivatives of $f(x)$ is $F(x) = x^2 + C$.
- Often times we will use the symbol $\int f(x)\,\mathrm{d}x$ to denote the class of antiderivatives of $f(x)$; for example,

$$\int 2x\,\mathrm{d}x = x^2 + C$$

Definite Integrals

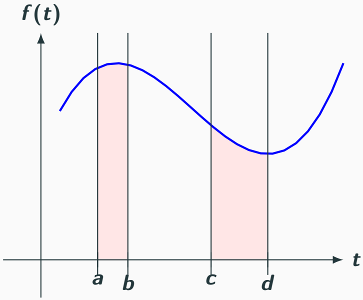
$$\int_a^b f(x)\ \mathrm{d}x = \text{Area under } f(x)$$



Definite Integrals

$$\int_R f(x)\,\mathrm{d}x = \text{Area under } f(x),$$

above the region R



$$R = [a, b) \cup (c, d)$$

Fundamental Theorem of Calculus

- Hold on; so we use $\int$ to denote both definite and indefinite integrals? Why?
- It turns out there is a very important link between definite and indefinite integrals!

Theorem: Fundamental Theorem of Calculus

(I) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{f(x)}^{g(x)} h(x)\ \mathrm{d}x = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$

(II) If $F(x)$ is an antiderivative of $f(x)$, $\displaystyle\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a)$

- **Some Food for Thought:** Does the FTC part (II) capture the uniqueness of the definite integral?

Example

- As an example: what is the area underneath the graph of the function $f(x) = \cos(x)$ between $x = 0$ and $x = \pi/2$?
- In other words, we seek

$$\int_0^{\pi/2} \cos(x)\ \mathrm{d}x$$

- We know that the primary antiderivative of $f(x) = \cos(x)$ is $F(x) = \sin(x)$. Therefore, by the Fundamental Theorem of Calculus,

$$\int_0^{\pi/2} \cos(x)\ \mathrm{d}x = \Big[\sin(x)\Big]_{x=0}^{x=\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = \boxed{1}$$

Integration by $u-$substitution

Derivation

- Start with the chain rule for derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x} f[g(x)] = f'[g(x)] \cdot g'(x)$$

- Now, integrate both sides with respect to x:

Formula: Integration by $u-$substitution

$$f[g(x)] = \int f'[g(x)] \cdot g'(x)\ \mathrm{d}x \tag{1}$$

- Often times, we will abbreviate $u := g(x)$ and $\mathrm{d}u = g'(x)\ \mathrm{d}x$.

Example 1

Suppose we wish to evaluate $\int e^{ay}\,\mathrm{d}y$ for some fixed $a > 0$.

- Set $u = ay$.
- Thus, $\mathrm{d}u = a\,\mathrm{d}y$, or equivalently, $\mathrm{d}y = {}^{1}\!/_{a}\,\mathrm{d}u$
- Returning to the integral:

$$\int e^{ay}\,\mathrm{d}y = \int e^{u} \cdot \frac{1}{a}\,\mathrm{d}u = \frac{1}{a}\int e^{u}\,\mathrm{d}u = \frac{1}{a}e^{u} + C$$

- Finally, convert back to y to see

$$\int e^{ay}\,\mathrm{d}y = \boxed{\frac{1}{a}e^{ay} + C}$$

Integration by Parts

Derivation

- Start with the product rule for derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

- Integrate both sides w.r.t. x:

$$f(x)g(x) = \int f'(x)g(x)\ \mathrm{d}x + \int f(x)g'(x)\ \mathrm{d}x$$

- Re-arrange terms to see

$$\int f(x)g'(x)\ \mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\ \mathrm{d}x$$

- Let $u := f(x)$ and $v := g(x)$ so $\mathrm{d}u = f'(x)\ \mathrm{d}x$ and $\mathrm{d}v = g'(x)\ \mathrm{d}x$, so we obtain

Formula: Integration by Parts

$$\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u \tag{2}$$

Example 2

Suppose we wish to evaluate $\int x\cos(x)\ \mathrm{d}x$

- Set $u = x$ and $\mathrm{d}v = \cos(x)\ \mathrm{d}x$.
- Then $\mathrm{d}u = \mathrm{d}x$ and $v = \int\cos(x)\ \mathrm{d}x = \sin(x)$
- Thus, by equation (2),

$$\int x\cos(x)\ \mathrm{d}x = x\sin(x) - \int \sin(x)\ \mathrm{d}x = \boxed{x\sin(x) + \cos(x) + C}$$

Example 3

Now, suppose we wish to evaluate $\int x^2 \cos(x)\, dx$

- Set $u = x^2$ and $dv = \cos(x)\, dx$.
- Then $du = 2x\, dx$ and $v = \int \cos(x)\, dx = \sin(x)$
- Thus, by equation (2),

$$\int x^2 \cos(x)\, dx = 2x \sin(x) - \int 2x \sin(x)\, dx$$

- To evaluate the integral on the RHS, we would need to use Integration by Parts again!
- What if we had to evaluate $\int x^5 \cos(x)\, dx$; then there would be *five* different integrations by parts!

Tabular Method for Integration by Parts

Illustration via Example

Let us return to $\int x^2 \cos(x)\, dx$.

- We still need to choose u and dv as above.
- Now, we begin constructing a table row by row.
- First write u on the left, and dv on the right:

x^2	$\cos(x)$

- Now, we move down one row.
- On the left, we differentiate; on the right, we integrate.

x^2	$\cos(x)$
$2x$	$\sin(x)$

Illustration via Example

- We continue until we reach 0 in the leftmost column:

x^2	$\cos(x)$
$2x$	$\sin(x)$
2	$-\cos(x)$
0	$-\sin(x)$

- Now, we connect diagonal terms, and place an alternating "+" and "−" sign above these connectors:

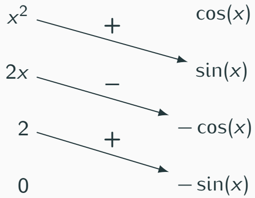


Illustration via Example

- Our final answer is obtained by multiplying along the connectors, and attaching the corresponding sign:

x^2	$+$	$\cos(x)$
$2x$	$-$	$\sin(x)$
2	$+$	$-\cos(x)$
0		$-\sin(x)$

$$\implies {}_{(+)}x^2\sin(x) + {}_{(-)}[-2x\cos(x)] + {}_{(+)}[-2\sin(x)]$$

or, after cleaning terms up,

$$x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

- You can verify (through differentiation) that this is indeed the correct answer!

Illustration via Example

- To see the benefit of this method, let us compute $\int x^5 \cos(x)\, dx$.

x^5		$\cos(x)$
	$+$	
$5x^4$		$\sin(x)$
	$-$	
$20x^3$		$-\cos(x)$
	$+$	
$60x^2$		$-\sin(x)$
	$-$	
$120x$		$\cos(x)$
	$+$	
120		$\sin(x)$
	$-$	
0		$-\cos(x)$

$$x^5 \sin(x) + 5x^4 \cos(x) - 20x^3 \sin(x) - 60x^2 \cos(x) + 120x \sin(x) + 120 \cos(x) + C$$

Integration by Inverse Trigonometric Substitution

Inverse Trig Sub

- I won't spend too much time deriving this one; please consult your Calculus Textbooks for a refresher.
- I will work through an example with you, though: say we want to evaluate

$$\int \sqrt{1 - x^2}\ \mathrm{d}x$$

- We let $x = \sin(\theta)$ so taht $\mathrm{d}x = \cos(\theta)\ \mathrm{d}\theta$; additionally, $1 - x^2 = 1 - \sin^2(\theta) = \cos^2(\theta)$, so

$$\int \sqrt{1 - x^2}\ \mathrm{d}x = \int \cos(\theta) \cdot \cos(\theta)\ \mathrm{d}\theta = \int \cos^2(\theta)\ \mathrm{d}\theta$$

- From here, we recall

$$\cos^2(\theta) = \frac{1}{2}\left[1 + \cos(2\theta)\right]$$

meaning

$$\int \cos^2(\theta)\ \mathrm{d}\theta = \int \frac{1}{2}\left[1 + \cos(2\theta)\right]\ \mathrm{d}\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$

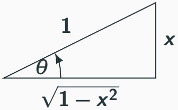
- Additionally,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

  meaning

$$\int \cos^2(\theta)\ d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C = \frac{1}{2}\left[\theta + \sin(\theta)\cos(\theta)\right] + C$$

- Our final task is to rewrite everything back in terms of x.
- Since $x = \sin(\theta)$, we see that $\theta = \arcsin(x)$. Additionally, since $\sin(\theta) = x$, we can draw a right triangle:



- From this triangle we see that $\cos(\theta) = \sqrt{1 - x^2}$.

Inverse Trig Sub

- Therefore, putting everything together:

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\left[\theta + \sin(\theta)\cos(\theta)\right] + C$$

$$= \frac{1}{2}\left[\arcsin(x) + x\sqrt{1-x^2}\right] + C$$

Partial Fraction Decomposition

Partial Fraction Decomposition

- Again, in the interest of time, I shall not rigorously define integration by Partial Fraction Decomposition, opting instead to work through an example.
- Suppose we wish to evaluate

$$\int \frac{1}{x^2 + 5x + 6} \, \mathrm{d}x$$

- First note that $x^2 + 5x + 6 = (x + 2)(x + 3)$. This means, by PFD (Partial Fraction Decomposition), the integrand can be written as

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

- Our goal is to find A and B. We do so by first bringing everything on the RHS to a single fraction:

$$\frac{1}{(x+2)(x+3)} = \frac{A(x+3) + B(x+2)}{(x+2)(x+3)} = \frac{x(A+B) + (3A+2B)}{(x+2)(x+3)}$$

- Matching terms, we find the following system of equations:

$$\begin{cases} A + B & = 0 \\ 3A + 2B & = 1 \end{cases}$$

Partial Fraction Decomposition

- We can now solve this system however we like; once you do so, you should find $A = 1$ and $B = -1$, meaning

$$\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$$

- Therefore, returning to our original integral,

$$\begin{aligned} \int \frac{1}{x^2+5x+6}\, \mathrm{d}x &= \int \left(\frac{1}{x+2} - \frac{1}{x+3} \right) \mathrm{d}x \\ &= \int \frac{1}{x+2}\, \mathrm{d}x - \int \frac{1}{x+3}\, \mathrm{d}x \\ &= \ln|x+2| - \ln|x+3| + C = \ln\left|\frac{x+2}{x+3}\right| + C \end{aligned}$$