Calculus Review: Sequences and Series

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Sequences

Sequences

- A **sequence** is a list of numbers, often indexed with a pattern.
- For example: $\{1, 2, 3, 4, 5\}$ is a **finite** sequence (since it has only a finite number of terms), where the k^{th} term is simply equal to k for $k \in \{1, 2, 3, 4, 5\}$.
- Another example: $\{2, 4, 6, 8, 10, 12, \cdots\}$. Here the "$\cdots$" mean this is a **infinite** sequence; the k^{th} term is simply $2k$ for any $k \in \mathbb{N}$.

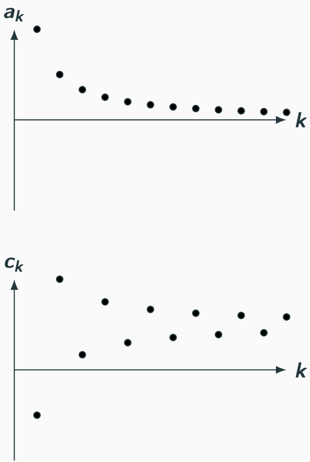
- In general, if the k^{th} term of a sequence is given by a_k, we write the sequence as

$$\{a_k\}_{k=1}^{\infty}$$

  or just $\{a_k\}$ for short.
  - For instance, the first sequence above can be notated $\{k\}_{k=1}^{5}$ and the second one can be notated $\{2k\}_{k=1}^{\infty}$

Sequences

- Here are some abstract examples of sequences:



- Qualitatively, we can see some things:

- As k gets large, both $\{a_k\}$ and $\{c_k\}$ seem to get arbitrarily close to some fixed number.
- This is not the case for $\{b_k\}$ or $\{d_k\}$. The sequence $\{b_k\}$ makes a beeline for ∞, whereas the sequence $\{d_k\}$ just keeps oscillating between two values.

- This leads us to the notion of **convergence**:

Definition

We say that a sequence $\{a_k\}$ **converges to the value L** if we can make a_k arbitrarily close to L. More mathematically:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})[k > N \implies |a_k - L| < \varepsilon]$$

Series

Partial Sums

- Consider a sequence $A := \{a_k\}_{k=1}^{\infty}$.
    - a_1 is a number.
    - $a_1 + a_2$ is a number.
    - $a_1 + a_2 + a_3$ is a number.
    - In general, $a_1 + \cdots + a_n$ for any fixed $n \in \mathbb{N}$ is a number.
- Therefore, let us consider the sequence S prescribed by:

$$S := \{a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \cdots\}$$

- The "pattern" of the sequence S is that S_n, the n^{th} element of S, is the sum of the first n elements of A:

$$S_n = a_1 + \cdots + a_n =: \sum_{k=1}^{n} a_k$$

- This sequence $S = \{S_n\}_{n=1}^{\infty}$ is called the sequence of **partial sums** of the original sequence a_k.

Partial Sums

- **Example:** What is the n^{th} partial sum of the sequence $\{k\}_{k=1}^{\infty}$?
- In other words, we seek a closed-form expression for $1 + 2 + \cdots + n$, for any fixed $n \in \mathbb{N}$.
- Here's the trick: let $S_n := 1 + \cdots + n$ denote the quantity we seek. Notice what happens when we add together two copies of S_n:

$$\begin{array}{rrcl} & S_n & = & 1 + 2 + 3 + \cdots + (n-1) + n \\ + & S_n & = & n + (n-1) + (n-2) + \cdots + 2 + 1 \\ \hline & 2S_n & = & (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) \end{array}$$

- On the LHS we have $2S_n$. On the RHS, we have n copies of $(n+1)$; in other words,

$$2S_n = n(n+1) \iff \boxed{S_n = \frac{n(n+1)}{2}}$$

- Using sigma notation, we have just shown that

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

Partial Sums

- **Example:** What is the n^{th} partial sum of the sequence $\{p^k\}_{k=0}^{\infty}$, where $p \in \mathbb{R}$ is a fixed constant?
- In other words, we seek a closed-form expression for $1 + p + p^2 + \cdots + p^n$, for any fixed $n \in \mathbb{N}$.
- Let $S_n := 1 + p + p^2 + \cdots + p^n$. Multiplying each term by p, we can see that

$$p \cdot S_n = p + p^2 + p^3 + p^4 + \cdots + p^n + p^{n+1}$$

- Let's see what happens when we subtract the second equation from the first:

$$\begin{array}{lrl} - & S_n & = 1 + p + p^2 + p^3 + \cdots + p^n \\ & p \cdot S_n & = \phantom{1 + {}} p + p^2 + p^3 + \cdots + p^n + p^{n+1} \\ \hline & S_n - p \cdot S_n & = 1 + 0 \ + 0 \ \ + 0 + \cdots + 0 - p^{n+1} \end{array}$$

- In other words, $(1 - p) \cdot S_n = 1 - p^{n+1}$ which implies that

$$S_n = \frac{1 - p^{n+1}}{1 - p}$$

- Using sigma notation, we have just shown that

$$\boxed{\sum_{k=0}^{n} p^k = \frac{1 - p^{n+1}}{1 - p}}$$

- Since the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ of a sequence $\{a_k\}_{k=1}^{\infty}$ is, well, a sequence, it makes sense to talk about its convergence.
- If S_n converges to some value S, we write

$$\sum_{k=1}^{\infty}(a_k) = S$$

- In other words,

$$\sum_{k=1}^{\infty}(a_k) = \lim_{n\to\infty}\left[\sum_{k=1}^{n} a_k\right]$$

- So, to evaluate an infinite summation, we first take the sum up to some fixed finite value n, find a closed-form expression for the resulting sum, and then let n go to ∞.

Geometric Series

- As an example, suppose we wish to determine whether or not $\sum_{k=0}^{\infty} p^k$ converges.
- We have already shown that

$$\sum_{k=0}^{n} p^k = \frac{1 - p^{n+1}}{1 - p}$$

so the question really boils down to whether or not the sequence

$$\frac{1 - p^{n+1}}{1 - p}$$

has a limit.

- There is a result that states $\lim_{n\to\infty}(p^n) = 0$ if $|p| < 1$; if $|p| > 1$ then $\{p^n\}_n$ diverges. Hence, our infinite series converges only when $|p| < 1$, in which case it evaluates to

$$\boxed{\sum_{n=0}^{\infty} p^n = \frac{1}{1 - p} \quad \text{if } |p| < 1}$$

- This is a **VERY** important result; it is called the **geometric series**.

Geometric Series

- As a more concrete example, consider $\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k$. Since $p = (1/2) < 1$, the series converges and

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{1}{1-\left(\frac{1}{2}\right)} = 2$$

- An extension of the Geometric Series states that

$$\boxed{\sum_{k=a}^{\infty} p^k = \frac{p^a}{1-p} \quad \text{if } |p| < 1}$$

Taylor and MacLaurin Expansions

Taylor and MacLaurin Expansions

- There exists a very powerful theorem from Calculus, which effectively states the following: given a "nice"[1] enough function $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1}$$

  for any $a \in \mathbb{R}$. The sum on the RHS is known as the **Taylor Series Expansion** (or simply the **Taylor Expansion**) of $f(x)$ about the point a.

- Setting $a = 0$ in equation (2) yields the so-called **MacLaurin Series Expansion** (aka **MacLaurin Expansion**) of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{2}$$

[1] For our purposes, "nice" can be considered synonymous with "infinitely differentiable"

Example: Exponential Series

Problem: Derive the MacLaurin Expansion of $f(x) = e^x$.

- Note that $f(x) = f'(x) = f''(x) = \cdots = e^x$; that is, $f^{(n)}(x) = e^x, \ \forall n \in \mathbb{N}$. This allows us to conclude that $f^{(n)}(0) = e^0 = 1$, and

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

- For example,

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2 \approx 7.3891$$

Manipulating Sums

Differentiation of Sums

- It turns out, under certain conditions (over which we will not concern ourselves for the purposes of this class), derivatives and infinite sums commute.
- As an example, consider the geometric series for a fixed common ratio $p < 1$:

$$\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$$

- Differentiating both sides w.r.t. p yields

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{k=0}^{\infty} p^k\right) = \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{1-p}\right) = \frac{1}{(1-p)^2}$$

- On the LHS, we can pass the derivative through the sum to obtain

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{k=0}^{\infty} p^n\right) = \sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}p}(p^k) = \sum_{k=1}^{\infty} kp^{k-1}$$

  - Note that after passing the derivative through, we started the sum from $k = 1$ instead of $k = 0$. This is because the first term of our original sequence is $p^0 = 1$, which doesn't depend on p: hence, when we take the derivative of that term w.r.t. p we simply get 0.

Differentiation of Sums

- Putting the pieces together, we find

$$\sum_{k=1}^{\infty} kp^{k-1} = \frac{1}{(1-p)^2}$$

  or, restarting the sum on the LHS from $k = 0$ (since $[kp^{k-1}]_{k=0} = 0$) and multiplying both sides by p,

$$\boxed{\sum_{k=0}^{\infty} kp^{k} = \frac{p}{(1-p)^2} \quad \text{if } |p| < 1}$$

- Perhaps this method will be useful on one of the Worksheet Problems for this week...

Integration of Sums

- Infinite sums can be integrated term-by-term as well!
- For example, let us again start with the geometric series assuming $|p| < 1$:

$$\frac{1}{1-p} = \sum_{k=0}^{\infty} p^k$$

- Replacing p with $(-p^2)$ yields

$$\frac{1}{1+p^2} = \sum_{k=0}^{\infty} (-1 \cdot p^2)^k = \sum_{k=0}^{\infty} (-1)^k p^{2k}$$

- Integrate both sides w.r.t. p:

$$\int \left(\frac{1}{1+p^2} \right) \, \mathrm{d}p = \int \left[\sum_{k=0}^{\infty} (-1)^k p^{2k} \right] \, \mathrm{d}p$$

- Once the dust settles, we have

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} p^{2k+1} = \arctan(p) \quad \text{if } |p| < 1}$$

Re-Indexing a Sum

- There's another tool I'd like to bring your attention to. I'll introduce this by way of an example.
- We already have a formula for the sum of the first n natural numbers. What if we seek a formula for the sum of only the *even* natural numbers, up to and including n? (For convenience, let's assume n itself is even.)
- In other words, we seek to evaluate

$$\sum_{\substack{k=2 \\ \text{even}}}^{n} (k)$$

- The trick is to note the following: any even number k can be written as $2m$, for another arbitrary integer m. Therefore, wherever I see a k I can replace it with $2m$, and then sum from $m = 1$ to $n/2$.

Re-Indexing a Sum

- Let me break that down a bit more for you.

$$\sum_{\substack{k=2\\\text{even}}}^{n}(k) = \underbrace{2}_{(m=1)} + \underbrace{4}_{(m=2)} + \underbrace{6}_{(m=3)} + \cdots + \underbrace{n}_{(m=\frac{n}{2})}$$

- So, I can write

$$\sum_{\substack{k=2\\\text{even}}}^{n}(k) = \sum_{m=1}^{\frac{n}{2}}(2m) = 2 \cdot \sum_{m=1}^{\frac{n}{2}}(m) = 2 \cdot \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2} = \frac{n}{2}\left(\frac{n}{2}+1\right)$$

- A similar trick holds for odd numbers; remember that any odd number k can be written as $(2m+1)$ for some natural number m.

- Re-indexing sums is very important!!!

To Summarize

- So, to summarize, here are some of the important sums we've learned (and these will continue to crop up throughout PSTAT 120A!):

  - $\displaystyle\sum_{k=1}^{n}(k) = \frac{n(n+1)}{2}$
  - $\displaystyle\sum_{k=0}^{n} p^k = \frac{1 - p^{n+1}}{1 - p}$
  - $\displaystyle\sum_{k=a}^{\infty} p^k = \frac{p^a}{1 - p}$, if $|p| < 1$
  - $\displaystyle\sum_{k=0}^{\infty} kp^k = \frac{p}{(1 - p)^2}$, if $|p| < 1$
  - $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$, for any $x \in \mathbb{R}$