HOMEWORK 1

PSTAT 120A: Summer 2022

Due: 11:59pm on Friday, June 24

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Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.
- Be sure to show your work; correct answers with no supporting work will not be awarded full points.
- 2 randomly selected questions will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

1. **Calculus Crash-Course.** Compute the following:

a) $\displaystyle\sum_{k=10}^{\infty} \frac{2}{3^k}$ **b** $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{4^k}{k!}$ **c)** $\displaystyle\sum_{\substack{k=4 \\ \text{even}}}^{\infty} \frac{2}{3^k}$

d) $\displaystyle\int_a^b xe^{-x^2}\,\mathrm{d}x$ **e)** $\displaystyle\int_a^b xe^{-x}\,\mathrm{d}x$ **f)** $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$

Solution:

(a) $$\sum_{k=10}^{\infty} \frac{2}{3^k} = 2\sum_{k=10}^{\infty}\left(\frac{1}{3}\right)^k = 2 \times \frac{\left(\frac{1}{3}\right)^{10}}{1-\frac{1}{3}} = \cancel{2} \times \frac{1}{3^{10}} \cdot \frac{3}{\cancel{2}} = \left(\frac{1}{3}\right)^9$$

(b) $$\sum_{k=1}^{\infty} (-1)^k \cdot \frac{4^k}{k!} = \sum_{k=1}^{\infty} \frac{(-4)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-4)^k}{k!} - \frac{(-4)^0}{0!} = e^{-4} - 1 = \frac{1-e^4}{e^4}$$

(c) $$\sum_{\substack{k=4 \\ \text{even}}}^{\infty} \frac{2}{3^k} = 2\sum_{\substack{k=4 \\ \text{even}}}^{\infty} \left(\frac{1}{3}\right)^k$$ Reindex: let $n := 2k$ so that

$$\begin{aligned}
\sum_{\substack{k=4 \\ \text{even}}}^{\infty} \frac{2}{3^k} &= 2\sum_{\substack{k=4 \\ \text{even}}}^{\infty} \left(\frac{1}{3}\right)^k \\
&= 2\sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^{2n} = 2\sum_{n=2}^{\infty} \left[\left(\frac{1}{3}\right)^2\right]^n = 2\sum_{n=2}^{\infty} \left(\frac{1}{9}\right)^n \\
&= 2 \times \frac{\left(\frac{1}{9}\right)^2}{1-\frac{1}{9}} = 2 \times \frac{1}{81} \times \frac{9}{8} = \frac{1}{36}
\end{aligned}$$

(d) Make a $u-$substitution: $u = x^2$ so that $\mathrm{d}u = 2x\,\mathrm{d}x$ and

$$\int_a^b xe^{-x^2}\,\mathrm{d}x = \int_{a^2}^{b^2} \frac{1}{2}e^{-u}\,\mathrm{d}u = \frac{1}{2}\left[-e^{-u}\right]_{u=a^2}^{u=b^2} = \frac{1}{2}\left(e^{-a^2} - e^{-b^2}\right)$$

(e) Integrate by parts with

$$\left\{ \begin{array}{ll} u = x & v = -e^{-x} \\ \mathrm{d}u = \mathrm{d}x & \mathrm{d}v = e^{-x}\,\mathrm{d}x \end{array} \right\}$$

to see

$$\int xe^{-x}\,\mathrm{d}x = -e^{-x}(x+1)\Big]_{x=a}^{x=b} = (a+1)e^{-a} - (b+1)e^{-b}$$

(f) We utilize an inverse trigonometric substitution: let $x = \sin(\theta)$ so that $\mathrm{d}x = \cos(\theta)\,\mathrm{d}\theta$, and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta)$$

(we can ignore the absolute values since x is constrained to be between 0 and 1, meaning $\cos(\theta)$ is constrained to be nonnegative). Then,

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1}{\cos(\theta)} \cdot \cos(\theta)\,\mathrm{d}\theta = \int \mathrm{d}\theta = \theta + C = \arcsin(x) + C$$

and so

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin(1) - \arcsin(0) = \frac{\pi}{2}$$

2. Refresher on Maclaurin Series Expansions

a) Consider the function $f(x) = \ln(1+x)$. Find an expression for $f^{(n)}(0)$, where $n \geq 0$ is an arbitrary integer. **Hint:** Your final answer should be piecewise-defined, with two cases.

Solution: Let's try writing out a few derivatives explicitly.

$$\begin{aligned} f^{(0)}(x) &= \ln(1+x) \\ f^{(1)}(x) &= \frac{1}{1+x} \\ f^{(2)}(x) &= -\frac{1}{(1+x)^2} \\ f^{(3)}(x) &= \frac{2 \cdot 1}{(1+x)^3} \\ f^{(4)}(x) &= -\frac{3 \cdot 2 \cdot 1}{(1+x)^4} \\ \vdots \quad & \quad \vdots \end{aligned}$$

Thus, we can see the following pattern emerging:

$$f^{(n)}(x) = \begin{cases} \ln(1+x) & \text{if } n = 0 \\ (-1)^{n-1} \cdot \frac{(n-1)!}{(1+x)^n} & \text{if } n \geq 1 \end{cases}$$

and so we have

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 0 \\ (n-1)!(-1)^{n-1} & \text{if } n \geq 1 \end{cases}$$

b) Using your answer from part (a), derive the Maclaurin Series Expansion of $f(x) = \ln(1+x)$.

Solution: In general, the MacLaurin Series Expansion of a function $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Therefore, by our expression from part (a), we have

$$\begin{aligned} \ln(1+x) &= f^{(0)}(0) + \sum_{n=1}^{\infty} \frac{(n-1)!(-1)^{n-1}}{n!} \\ &= 0 + \sum_{n=1}^{\infty} \frac{\cancel{(n-1)!} \cdot (-1)^{n-1}}{n \cdot \cancel{(n-1)!}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n \end{aligned}$$

c) Use your answer from part (b) to evaluate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Solution: Let's see what happens when we plug in $x = 1$ to our answer to part (b) above:

$$\ln(1+1) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

meaning

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln(1+1) = -\ln(2) = \ln\left(\frac{1}{2}\right)$$

3. A Simple Experiment. Suppose I toss a fair coin, roll a fair 4-sided die, and pick a number at random between 1 and 3 (inclusive), all at the same time.

a) Write down a possible outcome space Ω. Be sure to clearly define your notation!

Solution: Let H denote "the coin landed heads" and T denote "the coin landed tails"; let D_i denote "the die landed i" for $i = 1, 2, \dots, 6$ and let N_i denote "the number selected was i, for $i = 1, 2, 3$. Then our outcome space can be expressed as

$$\Omega = \{H, T\} \times \{D_1, D_2, D_3, D_4\} \times \{N_1, N_2, N_3\}$$

In other words, an outcome ω is a triple where the first entry denotes the result of the coin flip, the second denotes the result of the die roll, and the third denotes the result of the number. For instance, $\omega = (T, D_3, N_1)$ corresponds to the outcome "the coin landed tails, the die landed on the number 3, and the number selected was 1."

This is, of course, not the only notation we could have used. We could have simply used i to denote the result of the die and j to denote the result of the number selection, so that

$$\Omega' = \{H, T\} \times \{1, 2, 3, 4\} \times \{1, 2, 3\}$$

so that the outcome ω above can be expressed as $\omega = (T, 3, 1)$.

b) Does it make sense to utilize the classical definition of probability for this problem? Explain why or why not.

Solution: Yes, it makes sense to use the classical definition of probability since both the coin and die are fair and the number selected is selected at random.

c) Compute the probabilities of the following events, using the classical definition of probability:

(i) A is the event that the coin landed heads.

Solution: By part (b), we can take the classical definition of probability so that $\mathbb{P}(A) = |A|/|\Omega|$. We can see that $|\Omega| = 2 \cdot 4 \cdot 3 = 24$; additionally

$$A = \{H\} \times \{D_1, D_2, D_3, D_4\} \times \{N_1, N_2, N_3\}$$

meaning $|A| = 1 \times 4 \times 3 = 18$ and so

$$\mathbb{P}(A) = \frac{12}{24} = \frac{1}{2}$$

(ii) B is the event that the die shows a number strictly smaller than the number selected from $\{1, 2, 3\}$.

Solution: Our first task is to express the event B mathematically.

$$B = \underbrace{[\{H, T\} \times \{D_1\} \times \{N_2, N_3\}]}_{:=B_1} \cup \underbrace{[\{H, T\} \times \{D_1, D_2\} \times \{N_3\}]}_{:=B_2}$$

By the Addition Rule, $|B_1 \cup B_2| = |B_1| + |B_2| - |B_1 \cap B_2|$. We see that

$$|B_1 \cap B_2| = |\{H, T\} \times \{D_1\} \times \{N_3\}| = 2 \cdot 1 \cdot 1 = 2$$

and so

$$|B| = (2 \cdot 1 \cdot 2) + (2 \cdot 2 \cdot 1) - (2 \cdot 1 \cdot 1) = 8 - 2 = 6$$

meaning

$$\mathbb{P}(B) = \frac{6}{24} = \frac{1}{4}$$

There is another (perhaps faster) way to find $|B|$ as well, and that is to use the slot method. Here is the general thought process:

- Suppose the number selected were 1; then there are no possibilities for the result of the die roll, since we require the result of the die roll to be strictly less than the number selected. Therefore we disregard this case.
- Suppose the number selected were 2; then the die must have rolled 1, and the coin could have landed either heads or tails. Thus, our slot diagram looks like

  $$\underline{2} \quad \underline{1} \quad \underline{1}$$

  which, by the multiplication rule, corresponds to $2 \cdot 1 \cdot 1 = 2$ possibilities.
- Suppose the number selected were 3; then the die could have landed on either 1 or 2, and the coin could have landed either heads or tails. Thus, our slot diagram looks like

  $$\underline{2} \quad \underline{2} \quad \underline{1}$$

  which, by the multiplication rule, corresponds to $2 \cdot 2 \cdot 1 = 4$ possibilities.

Now, these three cases yield three mutually exclusive configurations; thus, by the Addition Rule, $|B| = 2 + 4 = 6$ which is precisely what we obtained above.

(iii) C is the event that the coin landed heads, <u>and</u> the die shows a number strictly smaller than the number selected from $\{1, 2, 3\}$.

Solution: We have

$$C = \underbrace{[\{H\} \cap \{D_1\} \cap \{N_2, N_3\}]}_{:=C_1} \cup \underbrace{[\{H\} \cap \{D_1, D_2\} \cap \{N_3\}]}_{:=C_2}$$

Additionally,

$$C_1 \cap C_2 = \{(H, D_1, N_3)\} \implies |C_1 \cap C_2| = 1$$

which shows

$$|C| = (1 \cdot 1 \cdot 2) + (1 \cdot 2 \cdot 1) - 1 = 4 - 1 = 3$$

and so

$$\mathbb{P}(C) = \frac{3}{24} = \frac{1}{8}$$

We could have also used the slot method again:

- If the number selected were 1, we still have no possibilities for the die roll.
- If the number selected were 2, the die must have rolled 1 and our slot diagram looks like

$$\underline{1} \quad \underline{1} \quad \underline{1}$$

- If the number selected were 2, then the die could have landed on either 1 or 2 so our slot diagram looks like

$$\underline{1} \quad \underline{2} \quad \underline{1}$$

Thus

$$|C| = (1 \cdot 1 \cdot 1) + (1 \cdot 2 \cdot 1) = 3$$

which is exactly what we obtained above.

(iv) D is the event that the coin landed heads, or the die shows a number strictly smaller than the number selected from $\{1, 2, 3\}$.

Solution: Note that

$$\mathbb{P}(D) = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

where A and B are defined as in parts (i) and (ii) above. Additionally, $\mathbb{P}(A \cap B) = \mathbb{P}(C)$, where C is as defined in part (iii) above; hence

$$\mathbb{P}(D) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(C) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$$

4. **Counting Students.** In a particular section of PSTAT 120A, there are 100 students: 30 Freshman, 40 Sophomores , 20 Juniors, and 10 Seniors. Of the freshman, 20 are PSTAT majors; of the Sophomores, 10 are PSTAT majors; of the Juniors, 5 are PSTAT Majors; and of the Seniors, 2 are PSTAT Majors. A random subset of 10 of these students is to be selected.

(a) What is the probability that this sample contains only Freshman?

Solution:

- From the 30 total freshmen, we wish to select 10; this contributes a factor of $\binom{30}{5}$ to our answer.
- From the remaining $100 - 30 = 70$ students, we wish to select none.

Therefore, dividing by $\binom{100}{10}$ (which is the total number of ways to select 10 students from a total of 100), we find that our final probability is

$$\boxed{\frac{\binom{30}{10}}{\binom{100}{10}}}$$

(b) ~~What is the probability that this sample contains at least one student from each cohort (Freshman, Sophomore, Junior, Senior)?~~

(c) What is the probability that this sample contains only PSTAT Majors?

Solution: We do not care about the cohorts; as such, we can consider grouping all of the PSTAT students into one group, and then selecting 10 from these. There are a total of $20 + 10 + 5 + 2 = 37$ PSTAT students; hence, our final probability is

$$\boxed{\frac{\binom{37}{10}}{\binom{100}{10}}}$$

(d) Let A denote the event "the sample contains only Freshman" and B denote the event "the sample contains 5 PSTAT Majors." Compute $\mathbb{P}(A \cap B)$.

Solution: Note that $(A \cap B)$ means "all ten students were freshmen, and 5 were PSTAT Majors." In other words, we wish to pick 5 students from the 20 that are PSTAT Freshmen, and an additional 5 from the remaining $30 - 20 = 10$ non-PSTAT Freshmen. Hence, the desired probability is

$$\boxed{\frac{\binom{20}{5}\binom{10}{5}}{\binom{100}{10}}}$$