HOMEWORK 2

PSTAT 120A: Summer 2022

Due: 11:59pm on Tuesday, June 28

Instructor: Ethan P. Marzban

Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.
- Be sure to show your work; correct answers with no supporting work will not be awarded full points.
- 2 randomly selected questions/parts will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

1. In a given population, 5% are infected with a particular disease. There exists a test for this disease, but it is fairly inaccurate. Of those who are healthy, the test correctly identifies them as disease-free 20% of the time, but incorrectly identifies them as disease-positive 40% of the time; the remaining 40% of the time, the test simply returns a result of "Inconclusive." Similarly, of those who are diseased, the test correctly identifies them as disease-positive 30% of the time, but incorrectly identifies them as disease-free 50% of the time; the remaining 20% of the time, the test returns a result of "Inconclusive."

   John has taken a test, and the test has returned a result of "Inconclusive." What is the probability that John actually has the disease?

2. The Celestial Toymaker has decided to play a game with me. On a table, he lines up an infinite number of boxes (he *is* magical, after all). With probability $(1/2)^i$ he selects box number i [where $i = 1, 2, 3, \dots$]. Inside box number i there are 3^i marbles, one of which is red and the remainder of which are blue. So, for example, box 1 is selected with probability (1/2), and contains 1 red marble and 2 blue marbles; box 2 is selected with probability (1/4), and contains 1 red marble and 8 blue marbles, etc. Bumi selects a box, and then draws a marble.
   (a) What is the probability that the Toymaker selects a red marble?
   (b) Given that the Toymaker selected a red marble, what is the problem that he drew from box 4?

3. Alicia, Barbara, and Cassandra each draw a ticket at random from a box containing tickets labelled 1 through 100. (Assume that each person replaces the ticket they have drawn, so that it is possible for two or more people to draw the same number.) Define the following events:

$$\begin{aligned} A &:= \{\text{Alicia and Barbara drew the same number}\} \\ B &:= \{\text{Barbara and Cassandra drew the same number}\} \\ C &:= \{\text{Cassandra and Alicia drew the same number}\} \end{aligned}$$

   (a) Are A, B, and C pairwise independent? Justify your answer.
   (b) Are A, B, and C independent? Justify your answer.

4. A random variable X has state space $S_X = \{0, 1, 2, \cdots\}$ and probability mass function (p.m.f.) given by

$$\mathbb{P}(X = k) = \begin{cases} c \cdot \dfrac{3^k}{k!} & \text{if } x \in \{0, 1, 2, \cdots\} \\ 0 & \text{otherwise} \end{cases}$$

   where $c > 0$ is an as-of-yet undetermined constant.
   (a) Find the value of c that ensures $\mathbb{P}(X = k)$ is a valid probability mass function (p.m.f.).
   (b) Compute $\mathbb{P}(X \geq 3)$. Do not leave any infinite sums unsimplified!
   (c) Compute $\mathbb{E}[X]$. Do not leave any infinite sums unsimplified!
   (d) Compute $\mathbb{E}\left[\frac{X!}{5^X}\right]$. Do not leave any infinite sums unsimplified!