HOMEWORK 2

PSTAT 120A: Summer 2022

Due: 11:59pm on Tuesday, June 28

Instructor: Ethan P. Marzban

Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.
- Be sure to show your work; correct answers with no supporting work will not be awarded full points.
- 2 randomly selected questions/parts will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

1. In a given population, 5% are infected with a particular disease. There exists a test for this disease, but it is fairly inaccurate. Of those who are healthy, the test correctly identifies them as disease-free 20% of the time, but incorrectly identifies them as disease-positive 40% of the time; the remaining 40% of the time, the test simply returns a result of "Inconclusive." Similarly, of those who are diseased, the test correctly identifies them as disease-positive 30% of the time, but incorrectly identifies them as disease-free 50% of the time; the remaining 20% of the time, the test returns a result of "Inconclusive."

   John has taken a test, and the test has returned a result of "Inconclusive." What is the probability that John actually has the disease?

Solution: Let's establish some notation first.

$$\begin{aligned} D &= \text{"John has the disease"} \\ P &= \text{"John tests posiive"} \\ N &= \text{"John tests negative"} \\ I &= \text{"John tests inconclusive"} \end{aligned}$$

In this notation, we can translate the information provided in the problem:

$$\begin{aligned} \mathbb{P}(D) &= 0.05 \\ \mathbb{P}(N \mid D^{\complement}) &= 0.2 \\ \mathbb{P}(P \mid D^{\complement}) &= 0.4 \\ \mathbb{P}(I \mid D^{\complement}) &= 0.4 \\ \mathbb{P}(P \mid D) &= 0.3 \\ \mathbb{P}(N \mid D) &= 0.5 \\ \mathbb{P}(I \mid D) &= 0.2 \end{aligned}$$

Now, we seek $\mathbb{P}(D \mid I)$; by Bayes Rule, we compute this as

$$\mathbb{P}(D \mid I) = \frac{\mathbb{P}(I \mid D)\mathbb{P}(D)}{\mathbb{P}(I)}$$

The numerator can be computed directly, using quantities provided in the problem. For the denominator, we apply the Law of Total Probability:

$$\begin{aligned} \mathbb{P}(I) &= \mathbb{P}(I \mid D)\mathbb{P}(D) + \mathbb{P}(I \mid D^{\complement})\mathbb{P}(D^{\complement}) \\ &= (0.2)(0.05) + (0.4)(0.95) \end{aligned}$$

Therefore, putting everything together, we find

$$\begin{aligned}\mathbb{P}(D \mid I) &= \frac{\mathbb{P}(I \mid D)\mathbb{P}(D)}{\mathbb{P}(I)} \\ &= \frac{(0.2)(0.05)}{(0.2)(0.05) + (0.4)(0.95)} \approx 0.0256 = 2.56\%\end{aligned}$$

2. The Celestial Toymaker has decided to play a game with me. On a table, he lines up an infinite number of boxes (he *is* magical, after all). With probability $(1/2)^i$ he selects box number i [where $i = 1, 2, 3, \dots$]. Inside box number i there are 3^i marbles, one of which is red and the remainder of which are blue. So, for example, box 1 is selected with probability (1/2), and contains 1 red marble and 3 blue marbles; box 2 is selected with probability (1/4), and contains 1 red marble and 9 blue marbles, etc. Bumi selects a box, and then draws a marble.

   (a) What is the probability that the Toymaker selects a red marble?

   Solution: Let B_i denote the event that box i was chosen, and let R denote the event that a red marble was chosen. From the problem statement, we have that

$$\mathbb{P}(B_i) = \left(\frac{1}{2}\right)^i; \qquad \mathbb{P}(R \mid B_i) = \frac{1}{3^i}$$

   We seek $\mathbb{P}(R)$; using the Law of Total Probability, we compute this as

$$\begin{aligned}\mathbb{P}(R) &= \sum_{i=1}^{\infty} \mathbb{P}(R \mid B_i) \cdot \mathbb{P}(B_i) \\ &= \sum_{i=1}^{\infty} \left(\frac{1}{3^i}\right) \cdot \left(\frac{1}{2}\right)^i = \sum_{i=1}^{\infty} \left(\frac{1}{6}\right)^i \\ &= \frac{\left(\frac{1}{6}\right)}{1 - \frac{1}{6}} = \frac{1}{6} \cdot \frac{6}{5} = \frac{1}{5}\end{aligned}$$

   (b) Given that the Toymaker selected a red marble, what is the problem that he drew from box 4?

   Solution: Using our notation from part (a), we seek $\mathbb{P}(B_4 \mid R)$. By Bayes' Rule, we compute this as

$$\begin{aligned}\mathbb{P}(B_4 \mid R) &= \frac{\mathbb{P}(R \mid B_4) \cdot \mathbb{P}(B_4)}{\mathbb{P}(R)} \\ &= \frac{\left(\frac{1}{3^4}\right) \cdot \left(\frac{1}{2^4}\right)}{\left(\frac{1}{5}\right)} = \frac{5}{1296} \approx 3.856 \times 10^{-3}\end{aligned}$$

3. Alicia, Barbara, and Cassandra each draw a ticket at random from a box containing tickets labelled 1 through 100. (Assume that each person replaces the ticket they have drawn, so that it is possible for

two or more people to draw the same number.) Define the following events:

$$\begin{aligned} A &:= \{\text{Alicia and Barbara drew the same number}\} \\ B &:= \{\text{Barbara and Cassandra drew the same number}\} \\ C &:= \{\text{Cassandra and Alicia drew the same number}\} \end{aligned}$$

(a) Are A, B, and C pairwise independent? Justify your answer.

Solution: Let (x, y, z) denote the outcome "Alicia drew x, Barbara drew y, and Cassandra drew z." Then we can express the outcome space as

$$\Omega = [|1:100|]^3 := \{(x, y, z) : x \in [|1:100|],\ y \in [|1:100|],\ z \in [|1:100|],\ \}$$

Additionally, since selection of the ticket is done at random we can utilize the Classical Definition of Probability to see

$$\mathbb{P}(A) = \frac{100 \cdot 1 \cdot 100}{100^3} = \frac{1}{100}$$

(the numerator is effectively a slot-method-type argument; Alicia has 100 numbers to choose from, but then Barbara has only 1 since we require her to select the same number as Alicia, and Cassandra is free to draw any of the 100 numbers.)

Why do we not multiply this by 2? Well, the outcome $(1, 1, 20)$ is included only once in our outcome space so multiplying by 2 will overcount.

Extending this logic to B and C reveal that they have the same probability:

$$\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{100}$$

- $A \cap B$ denotes the event "all three people selected the same number." Utilizing a slot-method-type argument, we see $|A \cap B| = 100 \cdot 1 \cdot 1$, meaning

$$\mathbb{P}(A \cap B) = \frac{100}{100^3} = \frac{1}{100^2} = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- Symmetry reveals that the same is true about $|B \cap C|$ and $|C \cap B|$.

Therefore, A, B, C **are** pairwise independent.

(b) Are A, B, and C independent? Justify your answer.

Solution: Notice that $A \cap B \cap C = A \cap B$, meaning

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \cap B) = \frac{1}{100^2} \neq \frac{1}{100^3} = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$$

Therefore, A, B, C **are not** independent.

4. A random variable X has state space $S_X = \{0, 1, 2, \cdots\}$ and probability mass function (p.m.f.) given by

$$\mathbb{P}(X = k) = \begin{cases} c \cdot \dfrac{3^k}{k!} & \text{if } x \in \{0, 1, 2, \cdots\} \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant.

(a) Find the value of c that ensures $\mathbb{P}(X = k)$ is a valid probability mass function (p.m.f.).

Solution: We require

(1) $\mathbb{P}(X = k)$ for every $k \in \mathbb{R}$

(2) $\sum_k \mathbb{P}(X = k) = 1$

For the first condition, note that $(3^k)/k! > 0$ for every $k \in \{0, 1, 2, \cdots\}$ meaning $\mathbb{P}(X = k) \geq 0$ for every k. For the second, we compute

$$\sum_{k=0}^{\infty} c\frac{3^k}{k!} = c \cdot \sum_{k=0}^{\infty} \frac{3^k}{k!} = c \cdot e^3$$

Setting this equal to 1 yields $k = e^{-3}$.

(b) Compute $\mathbb{P}(X \geq 3)$. Do not leave any infinite sums unsimplified!

Solution: We could write

$$\mathbb{P}(X \geq 3) = \sum_{k=3}^{\infty} \mathbb{P}(X = 3) = \sum_{k=3}^{\infty} e^{-3} \cdot \frac{3^k}{k!}$$

But this is an infinite series! It is much easier to use the complement rule to write

$$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X \leq 2) = 1 - \sum_{k=0}^{2} e^{-3} \frac{3^k}{k!}$$

(c) Compute $\mathbb{E}[X]$. Do not leave any infinite sums unsimplified!

Solution: Utilizing the definition of expectation we find

$$\begin{aligned}
\mathbb{E}[X] &:= \sum_k k \cdot \mathbb{P}(X = k) = \sum_{k=0}^{\infty} k \cdot e^{-3} \cdot \frac{3^k}{k!} \\
&= e^{-3} \sum_{k=1}^{\infty} k \cdot \frac{3^k}{k!} \\
&= e^{-3} \sum_{k=1}^{\infty} \frac{3^k}{(k-1)!} \\
&= 3e^{-3} \sum_{k=1}^{\infty} \frac{3^{k-1}}{(k-1)!} = 3\cancel{e^{-3}} \cdot \cancel{e^3} = 3
\end{aligned}$$

(d) Compute $\mathbb{E}\left[\frac{X!}{5^X}\right]$. Do not leave any infinite sums unsimplified!

Solution: We have the following formula:

$$\mathbb{E}[g(X)] = \sum_k g(k) \mathbb{P}(X = k)$$

Hence, plugging in $\mathbb{P}(X = k)$ as above and $g(k) = k!/(5^k)$ we find

$$\mathbb{E}\left[\frac{X!}{5^X}\right] = \sum_{k=0}^{\infty} \frac{\cancel{k!}}{5^k} \cdot e^{-3} \frac{3^k}{\cancel{k!}} = e^{-3} \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k = e^{-3} \cdot \frac{1}{1 - \left(\frac{3}{5}\right)} = \frac{5}{2} e^{-3}$$