Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.

- Be sure to show your work; correct answers with no supporting work will not be awarded full points.

- 2 randomly selected questions/parts will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

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1. Let X be a random variable with probability mass function (p.m.f.) given by

$$p_X(k) = \begin{cases} \dfrac{1}{k(k+1)} & \text{if } k = 1, 2, 3, \cdots \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that $p_X(k)$ is a valid p.m.f. Be sure to show **all** of your work; don't use WolframAlpha! **Hint:** Partial Fraction Decomposition.

   (b) Compute $\mathbb{E}[X]$.

   (c) Compute $\mathbb{E}[X^2]$. Use this to compute $\text{Var}(X)$.

   (d) Find $F_X(x)$, the cumulative mass function (c.m.f.) of X. **Hint:** You can reuse some of the work you did in part (a).

2. A recent survey has revealed that 30% of cars are Electric Vehicles (EV's), 30% are Hybrid, and the remaining 40% are gas. A surveyor goes from parking lot to parking lot, recording the status (i.e. EV, Hybrid, or Gas) cars one at a time. The surveyor, however, is a bit forgetful, and sometimes records the same car twice.

   (a) What is the probability that, among the first 10 cars surveyed, the surveyor observes exactly 3 EV's?

   (b) What is the probability that the 10th car the surveyor examines is her first Hybrid vehicle?

   (c) What is the probability that the 24th car the surveyor examines is her fourth Gas car?

   (d) On average, how many cars (including the final "successful" car) does the surveyor need to examine before observing her fourth Gas car?

   (e) What is the variance of the number of Hybrid vehicles the surveyor observes among the first 30 cars she examines?

3. Chen-7 is a highly contagious disease that is capable of killing a Time Lord in under a day. Thankfully, only 7% of the population of Gallifrey has been infected (so far...). Additionally, there exists a test for Chen-7 but it is imperfect: 80% of the time it correctly identifies a healthy Time Lord as disease-free, but 30% of the time it incorrectly identifies a diseased Time Lord as disease-free.

   The Doctor is worried that they may have contracted Chen-7, so they takes 20 independent tests. Of these 20 tests, 10 indicate that they have the disease and the remaining 10 indicate that they do not. With this information, what is the probability that the Doctor actually has Chen-7?

4. Let X be a discrete random variable with support contained entirely within the positive real line (such a random variable is said to be "nonnegative"). Prove the so-called **tail-sum formula**, which states

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k)$$

Here is a rough guide to help you through your proof:

- Write $\mathbb{P}(X \geq k)$ as a sum, thereby obtaining a double-sum on the RHS of the Tail-Sum formula.
- Change the order of summation (a dot diagram may be helpful)
- Show that the resulting summation, with the order of summation reversed, is in fact equal to $\mathbb{E}[X]$.