HOMEWORK 4

PSTAT 120A: Summer 2022

Due: 11:59pm on Tuesday, July 5

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Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.
- Be sure to show your work; correct answers with no supporting work will not be awarded full points.
- 2 randomly selected questions/parts will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

1. Consider a random variable X with probability density function (p.d.f.) given by

$$f_X(x) = \begin{cases} \frac{c}{x^3} & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant.

(a) Find the value of c that ensures the function $f_X(x)$ is a valid p.d.f.

Solution: In order for $f_X(x)$ to be a valid p.d.f., it must be nonnegative everywhere and integrate to 1. Clearly $x^3 \geq 0$ whenever $x \geq 0$, meaning $f_X(x)$ trivially satisfies the nonnegativity condition. For the integrating-to-unity condition, we compute

$$\int_{-\infty}^{\infty} f_X(x)\, \mathrm{d}x = \int_2^{\infty} \frac{c}{x^3}\, \mathrm{d}x = -\frac{c}{2}\left[\frac{1}{x^2}\right]_{x=2}^{x=\infty} = \frac{c}{8} \overset{!}{=} 1 \implies \boxed{c = 8}$$

(b) Compute $\mathbb{P}(X \geq 5 \mid X \geq 3)$.

Solution: By the definition of conditional probabilities,

$$\mathbb{P}(X \geq 5 \mid X \geq 3) = \frac{\mathbb{P}(\{X \geq 5\} \cap \{X \geq 3\})}{\mathbb{P}(X \geq 3)}$$

We can compute the denominator by simply integrating the joint. For the numerator, note that $\{X \geq 5\} \subseteq \{X \geq 3\}$ (in other words: if a number is greater than 5 it is automatically greater than 3) meaning

$$\{X \geq 5\} \cap \{X \geq 3\} = \{X \geq 5\}$$

Therefore, we compute

$$\mathbb{P}(X \geq 3) = \int_3^{\infty} \frac{8}{x^3}\, \mathrm{d}x = -4\left[\frac{1}{x^2}\right]_{x=3}^{x=\infty} = \frac{4}{9}$$

$$\mathbb{P}(X \geq 5) = \int_5^{\infty} \frac{8}{x^3}\, \mathrm{d}x = -4\left[\frac{1}{x^2}\right]_{x=5}^{x=\infty} = \frac{4}{25}$$

and so

$$\mathbb{P}(X \geq 5 \mid X \geq 3) = \frac{\mathbb{P}(\{X \geq 5\} \cap \{X \geq 3\})}{\mathbb{P}(X \geq 3)} = \frac{\mathbb{P}(X \geq 5)}{\mathbb{P}(X \geq 3)} = \frac{\left(\frac{4}{25}\right)}{\left(\frac{4}{9}\right)} = \boxed{\frac{9}{25}}$$

(c) Compute $\mathbb{E}[X]$

Solution: Definitionally,

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} x f_X(x) \, \mathrm{d}x = \int_2^{\infty} x \cdot \frac{8}{x^3} \, \mathrm{d}x = 8 \cdot \int_2^{\infty} \frac{1}{x^2} \, \mathrm{d}x = -8 \left[\frac{1}{x} \right]_{x=2}^{x=\infty} = 4$$

(d) Compute $\mathrm{Var}(X)$

Solution: By the LOTUS,

$$\mathbb{E}[X^2] := \int_{-\infty}^{\infty} x^2 f_X(x) \, \mathrm{d}x = \int_2^{\infty} x^2 \cdot \frac{8}{x^3} \, \mathrm{d}x = 8 \cdot \int_2^{\infty} \frac{1}{x} \, \mathrm{d}x = \infty \implies \mathrm{Var}(X) = \infty$$

(e) Find $F_X(x)$, the cumulative distribution function (c.d.f.) of X.

Solution: There are two cases to consider:

- If $x < 2$, then $F_X(x) = \int_{-\infty}^{x} f_X(x) \, \mathrm{d}x = \int_{-\infty}^{x} (0) \, \mathrm{d}x = 0$
- If $x \geq 2$, then

$$F_X(x) = \int_2^x \frac{8}{t^3} \, \mathrm{d}t = -4 \left[\frac{1}{t^2} \right]_{t=2}^{t=x} = -4 \left(\frac{1}{x^2} - \frac{1}{4} \right) = 1 - \frac{4}{x^2}$$

So, putting everything together,

$$F_X(x) = \begin{cases} 0 & \text{if } x < 2 \\ 1 - \frac{4}{x^2} & \text{if } x \geq 2 \end{cases}$$

(f) Find the 72nd percentile of the distribution of X.

Solution: Firstly, we know that the 2nd percentile will be greater than 2; thus, we need only to invert the portion of the c.d.f. that is valid for $x \geq 2$. Recall: to invert a function $y = f(x)$ we swap the places of x and y, and then re-solve for y in terms of x. So, we wish to solve

$$x = 1 - \frac{4}{y^2} \implies \frac{4}{y^2} = 1 - x \implies y^2 = \frac{4}{1 - x} \implies y = F_X^{-1}(x) = \frac{2}{\sqrt{1 - x}}$$

Therefore, the 72nd percentile is given by

$$F_X^{-1}(0.72) = \frac{2}{\sqrt{1 - 0.72}} \approx 3.78$$

2. A company manufactures steel rods that are to be of length 15 meters. However, due to imperfections in the manufacturing process, the length of a rod is actually uniformly distributed between 13 meters and 17 meters.

(a) What is the probability that a randomly selected rod will be longer than 16 meters?

> **Solution:** Let X denote the length (in meters) of a randomly selected rod; then, by the problem statement, $X \sim \text{Unif}[13, 17]$, and so
>
> $$f_X(x) = \begin{cases} \frac{1}{17-13} & \text{if } x \in [13, 17] \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{4} & \text{if } x \in [13, 17] \\ 0 & \text{otherwise} \end{cases}$$
>
> Therefore,
>
> $$\mathbb{P}(X \geq 16) = \int_{16}^{17} \frac{1}{4} \, \mathrm{d}x = \boxed{\frac{1}{4}}$$

(b) A sample of 100 rods is taken with replacement, and the number of rods that are longer than 16 meters is recorded. What is the probability that this sample of 100 rods contains at least 63 rods that are longer than 16 meters? **You may leave your answer as an unsimplified finite sum.**

> **Solution:** Let Y denote the number of rods, in the sample of 100, that are longer than 16 meters. Then, Y follows a binomial distribution where "success" is characterized by finding a rod that is longer than 16 meters. We computed this probability to be $1/4$, in part (a), meaning
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> $$Y \sim \text{Bin}(100, 1/4)$$
>
> and so
>
> $$\mathbb{P}(Y \geq 63) = \boxed{\sum_{k=63}^{100} \binom{100}{k} \left(\frac{1}{4} \right)^k \left(\frac{3}{4} \right)^{100-k}}$$

3. On a particular stretch of Highway 101, the speed limit is listed as 65mph. In actuality, the speed of a randomly selected car is a random variable that follows the normal distribution with mean 60mph and standard deviation 4mph.

(a) What is the probability that a randomly selected car will be speeding? (Here, speeding just means "travelling at a speed greater than the speed limit") **Leave your answer in terms of $\Phi(\cdot)$, the standard normal c.d.f.**.

> **Solution:** Let X denote the speed of a randomly selected car traveling along Highway 101; then, from the problem statement, $X \sim \mathcal{N}(60, 16)$. We seek $\mathbb{P}(X \geq 65)$; to compute this, we utilize standardization:
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> $$\mathbb{P}(X \geq 65) = \mathbb{P}\left(\frac{X - 60}{4} \geq \frac{65 - 60}{4} \right) = \mathbb{P}\left(Z \geq \frac{5}{4} \right) = 1 - \mathbb{P}\left(Z \leq \frac{5}{4} \right) = \boxed{1 - \Phi\left(\frac{5}{4} \right)}$$
>
> where $Z := \frac{X-60}{4}$.

(b) Samantha's car is travelling at the 75^{th} percentile of speeds. How fast is Samantha travelling? **Leave your answers in terms of $\Phi^{-1}(\cdot)$, the inverse of the standard normal c.d.f.**.

> **Solution:** Let c denote the speed of Samantha's Car; we know then that $\mathbb{P}(X \leq c) = 0.75$ where X is defined as in part (a). We can expand the LHS using standardization:
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> $$\mathbb{P}(X \leq c) = \mathbb{P}\left(\frac{X - 60}{4} \leq \frac{c - 60}{4} \right) = \Phi\left(\frac{c - 60}{4} \right)$$

meaning c must solve the equation

$$\Phi\left(\frac{c-60}{4}\right) = 0.75$$

Applying $\Phi^{-1}(\cdot)$ to both sides and solving yields

$$c = 4 \cdot \Phi^{-1}(0.75) + 60$$

If you're curious, this amounts to roughly 63 mph.

4. Cars arrive at a traffic light according to a Poisson Point Process at a constant rate of 12 cars per hour.
   (a) What is the probability that 10 cars arrive at the traffic light in a given one-hour interval?

Solution: Let X denote the number of cars that arrive at the light in a one-hour interval; then $X \sim \text{Pois}(12 \cdot 1)$ and

$$\mathbb{P}(X = 10) = e^{-12} \cdot \frac{12^{10}}{10!}$$

   (b) What is the probability that 5 cars arrive at the traffic light in a given 30-minute interval?

Solution: Let Y denote the number of cars that arrive at the light in a 30-minute interval; then $Y \sim \text{Pois}(12 \cdot 0.5)$ and

$$\mathbb{P}(Y = 5) = e^{-6} \cdot \frac{6^5}{5!}$$

   (c) What is the probability that the 4^{th} and 5^{th} cars arrive within 5 minutes of each other?

Solution: Let T denote the time **in hours** elapsed between the arrival of the 4^{th} and 5^{th} cars; then $T \sim \text{Exp}(12)$. Additionally, note that 5 minutes = (1/12) of an hour. Therefore, we seek $\mathbb{P}(T < 1/12)$, which can be computed by integrating the p.d.f. of the Exponential distribution:

$$\mathbb{P}(T < 1/12) = \int_0^{1/12} 12e^{-12t}\,\mathrm{d}t = 1 - e^{-12 \cdot (1/12)} = 1 - e^{-1} \approx 63.2\%$$

   (d) What is the expected amount of time (in hours) that we must wait between the arrival of the 1^{st} and 4^{th} cars?

Solution: Let S denote the amount of time (in hours) between the 1^{st} and 4^{th} arrivals; then $S \sim \text{Gamma}(3, 12)$ and so

$$\mathbb{E}[S] = \frac{3}{12} = \frac{1}{4}\text{ hr} = 25 \text{ minutes}$$