HOMEWORK 5

PSTAT 120A: Summer 2022

Due: 11:59pm on Wednesday, July 13
Instructor: Ethan P. Marzban

Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.
- Be sure to show your work; correct answers with no supporting work will not be awarded full points.
- 2 randomly selected questions/parts will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

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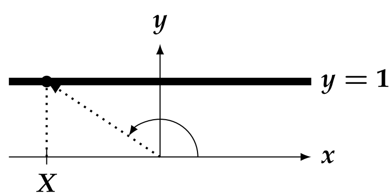
1. In each of the following parts you are supplied a random variable X with a provided probability density function (p.d.f.), along with a new random variable Y defined to be a particular function of X. Find the probability density function (p.d.f.) of Y. You may use either the c.d.f. method or the Change of Variable formula; just be sure to show all of your work. Additionally, be sure to specify the values over which your p.d.f. is nonzero.
   (a) $X \sim \text{Unif}[0, 2]$; $Y := X^2$
   (b) $X \sim \text{Unif}[-2, 2]$ $Y := X^2$
   (c) $X \sim \mathcal{N}(0, 1)$; $Y := e^X$. The distribution of Y is called the **Lognormal** distribution.
   (d) $X \sim \text{Exp}(\lambda)$; $Y := X^\beta$ for some fixed $\beta > 0$. The distribution of Y is called the **Weibull** distribution.

2. A particle is fired from the origin in a random direction pointing somewhere in the first two quadrants. The particle travels in a straight line, unobstructed, until it collides with an infinite wall located at $y = 1$. Let X denote the $x-$coordinate of the point of collision.



   (a) What is the expected value of the $x-$coordinate of the point of collision? **Do NOT first find the p.d.f. of X.**
   (b) Find $f_X(x)$, the probability density function (p.d.f.) of X
   (c) Confirm your answer to part (a) using your answer to part (b).

3. **Insurance Deductibles**. Here is a quick crash-course on how deductibles work. Suppose the insurance policy you purchased on your car comes with a $500 deductible. Then, if you get into an accident the amount you have to pay out-of-pocket follows the following scheme: if the true cost of damages is under $500 then you pay the full cost of damages, but if the true cost of damages is over $500 then you only pay $500 (and your insurance company pays the rest). So, if the true cost of damages is say $1,000 then you only pay $500.

   Suppose now that your deductible is m, where m is a fixed positive constant. Let X denote the true cost of damages of a particular accident, and let Y denote the amount of money you actually pay as a result of that accident. Further suppose that X is well-modeled by an $\text{Exp}(\lambda)$ distribution for some $\lambda > 0$.

(a) Express Y as a function of X. In other words, find an explicit formulation for the function $g(k)$ such that $Y = g(X)$.

(b) What is the expected amount of money you will have to pay?

(c) Find $F_Y(y)$, the cumulative distribution function (c.d.f.) of Y. **Two Hints:**
- Consider three cases: $y < 0,\ 0 \le y < m$, and $y > m$
- In each case, relate the event $\{Y \le y\}$ to something involving X
- Consider $\mathbb{P}(Y = m)$ separately.

(d) Is Y continuous, discrete, or neither?

4. **Double Integrals:** No plugging into WolframAlpha on this question; show **ALL** of your work!

(a) Compute $\displaystyle\int_0^1 \int_0^2 xy \; \mathrm{d}x \; \mathrm{d}y$

(b) Compute $\displaystyle\int_0^\infty \int_x^\infty e^{-y^2} \; \mathrm{d}y \; \mathrm{d}x$

(c) Compute $\displaystyle\iint_{\mathcal{R}} x^2y^2 \; \mathrm{d}A$ where $\mathcal{R}$ is the region $\mathcal{R} := \{(x, y) : |x| + |y| \le 1\}$