HOMEWORK 5

PSTAT 120A: Summer 2022

Due: 11:59pm on Wednesday, July 13

Instructor: Ethan P. Marzban

Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.
- Be sure to show your work; correct answers with no supporting work will not be awarded full points.
- 2 randomly selected questions/parts will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

1. In each of the following parts you are supplied a random variable X with a provided probability density function (p.d.f.), along with a new random variable Y defined to be a particular function of X. Find the probability density function (p.d.f.) of Y. You may use either the c.d.f. method or the Change of Variable formula; just be sure to show all of your work. Additionally, be sure to specify the values over which your p.d.f. is nonzero.

   (a) $X \sim \text{Unif}[0, 2]$; $Y := X^2$

   Solution: If we were to use the CDF method, we would write

   $$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

   Since $X \sim \text{Unif}[0, 2]$ we see that $F_X(-\sqrt{y}) = 0$, meaning

   $$F_Y(y) = F_X(\sqrt{y}) = \frac{\sqrt{y} - 0}{2 - 0} = \frac{\sqrt{y}}{2}$$

   and so, differentiating w.r.t. y,

   $$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\sqrt{y}}{2}\right) = \frac{1}{4\sqrt{y}}$$

   Additionally, $S_Y = [0, 4]$ and so

   $$f_Y(y) = \frac{1}{4\sqrt{y}} \cdot \mathbb{1}_{\{y \in [0,4]\}}$$

   (b) $X \sim \text{Unif}[-2, 2]$ $Y := X^2$

   Solution: Again, using the c.d.f. method:

   $$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(|X| \le \sqrt{y}) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

   Now, however, we cannot disregard the negative portion: if $y \in [0, 4]$ we have

   $$F_Y(y) = \frac{\sqrt{y}}{4} - \frac{-\sqrt{y}}{4} = \frac{\sqrt{y}}{2}$$

   meaning, differentiating w.r.t. y we find

   $$f_Y(y) = \frac{1}{4\sqrt{y}}$$

and so, putting everything together,

$$\boxed{f_Y(y) = \frac{1}{4\sqrt{y}} \cdot \mathbb{1}_{\{y \in [0,4]\}}}$$

(c) $X \sim \mathcal{N}(0,1)$; $Y := e^X$. The distribution of Y is called the **Lognormal** distribution.

Solution: Using the c.d.f. method we find

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(e^X \leq y) = \mathbb{P}(X \leq \ln y) = \Phi(\ln y)$$

Therefore, differentiating, we find

$$f_Y(y) = \frac{1}{y}\phi(\ln y) = \frac{1}{y} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\ln y)^2}$$

and, since $S_Y = [0, \infty)$ we have

$$\boxed{f_Y(y) = \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\ln y)^2} \cdot \mathbb{1}_{\{y \geq 0\}}}$$

(d) $X \sim \text{Exp}(\lambda)$; $Y := X^\beta$ for some fixed $\beta > 0$. The distribution of Y is called the **Weibull** distribution.

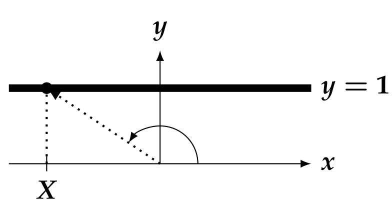
Solution: Now, though it is true that $g(t) = t^\beta$ is invertible only for some values of t, this is only true when we consider the real line; since the state space of X is $S_X = [0, \infty)$, the function $g(t) = t^\beta$ is always invertible over the domain of interest. Therefore, using the c.d.f. method, we find

$$F_Y(y) := \mathbb{P}(Y \leq y) = \mathbb{P}(X^\beta \leq y) = \mathbb{P}(X \leq \sqrt[\beta]{y}) = 1 - e^{-\lambda \sqrt[\beta]{y}}$$

and so, differentiating and incorporating the state space,

$$\boxed{f_Y(y) = \frac{\lambda}{\beta} \cdot y^{1/\beta - 1} \cdot e^{-\lambda y^{1/\beta}} \cdot \mathbb{1}_{\{y \geq 0\}}}$$

2. A particle is fired from the origin in a random direction pointing somewhere in the first two quadrants. The particle travels in a straight line, unobstructed, until it collides with an infinite wall located at $y = 1$. Let X denote the x−coordinate of the point of collision.



(a) What is the expected value of the x−coordinate of the point of collision? **Do NOT first find the p.d.f. of X.**

Solution: Let Θ denote the angle subtended by the trajectory of the particle, as measured from the positive $x-$axis. We can see then that

$$X = \cot(\Theta)$$

Since $\Theta \sim \text{Unif}[0, \pi]$ we can use the LOTUS to write

$$\begin{aligned}\mathbb{E}[X] = \mathbb{E}\left[\cot(\Theta)\right] &= \int_0^{\pi} \cot(\theta) \cdot \frac{1}{\pi} \, \mathrm{d}\theta \\ &= \frac{1}{\pi}\left[\int_0^{\pi/2} \cot(\theta) \, \mathrm{d}\theta + \int_{\pi/2}^{\pi} \cot(\theta) \, \mathrm{d}\theta\right]\end{aligned}$$

Let's focus on each integral separately.

$$\int_0^{\pi/2} \cot(\theta) \, \mathrm{d}\theta = \lim_{\beta \to 0} \int_{\beta}^{\pi/2} \cot(\theta) \, \mathrm{d}\theta = \lim_{\beta \to 0} \ln(\sin\theta)\Big|_{\theta=\beta}^{\theta=\pi/2} = \lim_{\beta \to 0} [0 - \ln(\sin\beta)] = -\infty$$

$$\int_{\pi/2}^{\pi} \cot(\theta) \, \mathrm{d}\theta = \lim_{\beta \to \pi} \int_{\pi/2}^{\beta} \cot(\theta) \, \mathrm{d}\theta = \lim_{\beta \to \pi} \ln(\sin\theta)\Big|_{\theta=\pi/2}^{\theta=\beta} = \lim_{\beta \to \pi} [\ln(\sin\beta)] = \infty$$

Therefore, we see that $\mathbb{E}[X]$ is undefined

(b) Find $f_X(x)$, the probability density function (p.d.f.) of X

Solution: Method 1: CDF Method For an $x \in \mathbb{R}$ we have

$$\begin{aligned}F_X(x) &:= \mathbb{P}(X \leq x) = \mathbb{P}(\cot\Theta \leq x) = \mathbb{P}\left(\Theta \geq \cot^{-1}(x)\right) = 1 - \frac{1}{\pi}\cot^{-1}(x) \\ f_X(x) &= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\pi}\cot^{-1}(x)\right) = \frac{1}{\pi(1+x^2)} \quad \text{for } x \in \mathbb{R}\end{aligned}$$

(Note that we flipped the sign of the inequality in the first line, since $\cot^{-1}(\cdot)$ is a monotonically decreasing function.)

Method 2: The Change of Variable Formula We take $g(t) = \cot(t)$ so that $g^{-1}(x) = \cot^{-1}(x)$ and

$$\left|\frac{\mathrm{d}}{\mathrm{d}x} g^{-1}(x)\right| = \left|\frac{1}{1+x^2}\right| = \frac{1}{1+x^2}$$

Since $f_\Theta(\theta) = {}^1/_\pi \cdot \mathbb{1}_{\{\theta \in [0,\pi]\}}$ we have

$$f_X(x) = \frac{1}{\pi} \cdot \mathbb{1}_{\{\cot^{-1}(\theta) \in [0,\pi]\}} \cdot \frac{1}{1+x^2} = \frac{1}{\pi(1+x^2)} \cdot \mathbb{1}_{\{x \in \mathbb{R}\}}$$

As an aside: This is a special case of what is known as the **Cauchy** distribution.

(c) Confirm your answer to part (a) using your answer to part (b).

Solution: We can see that

$$\int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\, \mathrm{d}x \text{ does not converge}$$

3. **Insurance Deductibles**. Here is a quick crash-course on how deductibles work. Suppose the insurance policy you purchased on your car comes with a \$500 deductible. Then, if you get into an accident the amount you have to pay out-of-pocket follows the following scheme: if the true cost of damages is under \$500 then you pay the full cost of damages, but if the true cost of damages is over \$500 then you only pay \$500 (and your insurance company pays the rest). So, if the true cost of damages is say \$1,000 then you only pay \$500.

   Suppose now that your deductible is m, where m is a fixed positive constant. Let X denote the true cost of damages of a particular accident, and let Y denote the amount of money you actually pay as a result of that accident. Further suppose that X is well-modeled by an $\text{Exp}(\lambda)$ distribution for some $\lambda > 0$.

   (a) Express Y as a function of X. In other words, find an explicit formulation for the function $g(k)$ such that $Y = g(X)$.

   Solution: From the problem statement, we can see

$$Y = \begin{cases} X & \text{if } X \leq m \\ m & \text{if } X > m \end{cases}$$

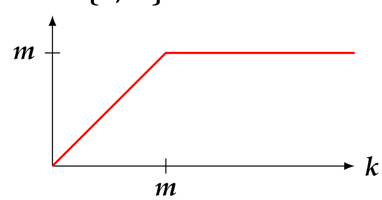
   If we wanted a slightly "neater" function, we can see

$$Y = \min\{X, m\}$$

   so

$$g(k) = \min\{k, m\} = \begin{cases} k & \text{if } k \leq m \\ m & \text{if } k \geq m \end{cases}$$

$g(k) = \min\{k, m\}$



   (b) What is the expected amount of money you will have to pay?

Solution: By the LOTUS,

$$\begin{aligned}
\mathbb{E}[Y] = \mathbb{E}[\min\{X, m\}] &= \int_{-\infty}^{\infty} \min\{x, m\} f_X(x) \, \mathrm{d}x \\
&= \int_{0}^{\infty} \min\{x, m\} \lambda e^{-\lambda x} \, \mathrm{d}x \\
&= \int_{0}^{m} x \lambda e^{-\lambda x} \, \mathrm{d}x + \int_{m}^{\infty} m \lambda e^{-\lambda x} \, \mathrm{d}x \\
&= \left[-e^{-\lambda x} \left(x + \frac{1}{\lambda} \right) \right]_{x=0}^{x=m} + m e^{-\lambda m} \\
&= \frac{1}{\lambda} - e^{-\lambda m} \left(m + \frac{1}{\lambda} \right) + m e^{-\lambda m} \\
&= \frac{1}{\lambda} - \cancel{m e^{-\lambda m}} - \frac{1}{\lambda} e^{-\lambda m} + \cancel{m e^{-\lambda m}} = \boxed{\frac{1}{\lambda} \left(1 - e^{-\lambda m} \right)}
\end{aligned}$$

(c) Find $F_Y(y)$, the cumulative distribution function (c.d.f.) of Y. **Two Hints:**
- Consider three cases: $y < 0$, $0 \leq y < m$, and $y > m$
- In each case, relate the event $\{Y \leq y\}$ to something involving X
- Consider $\mathbb{P}(Y = m)$ separately.

Solution:

- If $y < 0$, we see that $F_Y(y) = 0$ (i.e. we never pay a negative amount)
- If $y \geq m$ we see that $F_Y(y) = 1$ (i.e., we never pay more than \$$m$)
- If $0 < y < m$, we have that $\{Y \leq y\} = \{\min\{X, m\} \leq y\} = \{X \leq y\}$ (if the true amount of damages is less than m, the amount we pay is equal to the true amount of damages) meaning
$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X \leq y) = 1 - e^{-\lambda y}$$
- When computing $\mathbb{P}(Y = m)$, we note that $\{Y = m\} = \{X \geq m\}$ (since if the true amount of damages is greater than or equal to m, we only pay m.) That is, $\mathbb{P}(Y = m) = e^{-\lambda m}$ which is in fact consistent with what we have above.

Therefore:

$$\boxed{F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - e^{-\lambda y} & \text{if } 0 \leq y < m \\ 1 & \text{if } y \geq m \end{cases}}$$

(By the way, we do see that the c.d.f jumps a magnitude of $e^{-\lambda m}$ at $y = m$, consistent with our fourth point above)

(d) Is Y continuous, discrete, or neither?

Solution: We can see that the c.d.f. is not a step function, meaning it is not continuous. However, $F_Y(y)$ has a jump discontinuity at $y = m$; hence it is not strictly continuous either! Therefore, it is neither continuous nor discrete. More accurately, Y possesses both discrete and continuous aspects; this is an example of what we call a **mixed** distribution.

4. **Double Integrals:** No plugging into WolframAlpha on this question; show **ALL** of your work!

   (a) Compute $\int_0^1 \int_0^2 xy \, \mathrm{d}x \, \mathrm{d}y$

   Solution:

   $$\int_0^1 \int_0^2 xy \, \mathrm{d}x \, \mathrm{d}y = \left(\int_0^1 x \, \mathrm{d}x \right) \left(\int_0^2 y \, \mathrm{d}y \right) = \frac{1}{2} \cdot 2 = \boxed{1}$$

   (b) Compute $\int_0^\infty \int_x^\infty e^{-y^2} \, \mathrm{d}y \, \mathrm{d}x$

   Solution: We must reverse the order of integration in this problem:

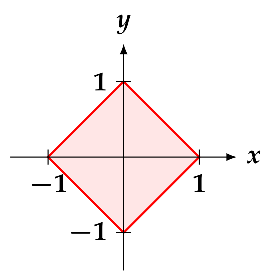
   $$\begin{aligned} \int_0^\infty \int_x^\infty e^{-y^2} \, \mathrm{d}y \, \mathrm{d}x &= \int_0^\infty \int_0^y e^{-y^2} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^\infty y e^{-y^2} \, \mathrm{d}y \end{aligned}$$

   Make a $u-$substitution: let $u = y^2$ so that $\mathrm{d}u = 2y \, \mathrm{d}y$ and so

   $$\begin{aligned} \int_0^\infty \int_x^\infty e^{-y^2} \, \mathrm{d}y \, \mathrm{d}x &= \int_0^\infty y e^{-y^2} \, \mathrm{d}y \\ &= \int_0^\infty \frac{1}{2} e^{-u} \, \mathrm{d}u = \boxed{\frac{1}{2}} \end{aligned}$$

   (c) Compute $\iint_{\mathcal{R}} x^2 y^2 \, \mathrm{d}A$ where $\mathcal{R}$ is the region $\mathcal{R} := \{(x, y) : |x| + |y| \leq 1\}$

   Solution: Let's sketch the region of integration:



   Either order of integration is fine; let's do $\mathrm{d}x \, \mathrm{d}y$.

   $$\begin{aligned} \iint_{\mathcal{R}} x^2 y^2 \, \mathrm{d}A &= \int_0^1 \int_{y-1}^{1-y} x^2 y^2 \, \mathrm{d}x \, \mathrm{d}y + \int_{-1}^0 \int_{-1-y}^{y+1} x^2 y^2 \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 \frac{1}{3} y^2 \left[(1-y)^3 - (y-1)^3 \right] \, \mathrm{d}y + \int_{-1}^0 \frac{1}{3} y^2 \left[(y+1)^3 - (-1-y)^3 \right] \, \mathrm{d}y \\ &= \int_0^1 \frac{1}{3} y^2 \left[2 - 6y + 6y^2 - 2y^3 \right] \, \mathrm{d}y + \int_{-1}^0 \frac{1}{3} y^2 \left[2 + 6y + 6y^2 + 2y^3 \right] \, \mathrm{d}y \\ &= \frac{1}{3} \left(\frac{2}{3} - \frac{6}{4} + \frac{6}{5} - \frac{2}{6} \right) + \frac{1}{3} \left[2 + 6y + 6y^2 + 2y^3 \right] = \boxed{\frac{2}{90}} \end{aligned}$$