HOMEWORK 6

PSTAT 120A: Summer 2022

Due: 11:59pm on Friday, July 15

Instructor: Ethan P. Marzban

Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.
- Be sure to show your work; correct answers with no supporting work will not be awarded full points.
- 2 randomly selected questions/parts will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

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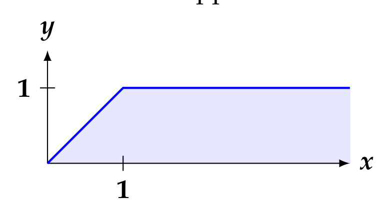
1. Let (X, Y) be a random vector with joint p.d.f. given by

$$f_{X,Y}(x,y) = \begin{cases} c \cdot \left(\frac{y}{x} \right)^4 & \text{if } (x,y) \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant, and $\mathcal{R}$ is the region in the first quadrant below the graph of $y = \min\{x, 1\}$.

(a) Find the value of c.

Solution: It will be useful to first sketch the support:

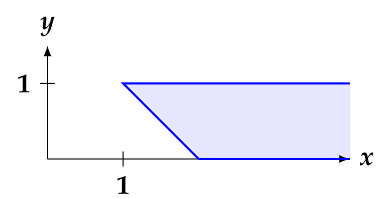


From this picture, we see that integrating in the order $\mathrm{d}x\ \mathrm{d}y$ will be simplest:

$$\begin{aligned} \iint_{\mathcal{R}} \left(\frac{y}{x} \right)^4 \mathrm{d}A &= \int_0^1 \int_y^\infty \frac{y^4}{x^4} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 y^4 \left[-\frac{1}{3x^3} \right]_{x=y}^{x=\infty} \mathrm{d}y \\ &= \frac{1}{3} \int_0^1 y^4 \cdot \frac{1}{y^3} \, \mathrm{d}y = \frac{1}{3} \int_0^1 y \, \mathrm{d}y = \frac{1}{6} \implies c = 6 \end{aligned}$$

(b) Set up, but do not evaluate, the double integral corresponding to $\mathbb{P}(X + Y \geq 2)$.

Solution: We first sketch the region of integration:



Once again, the order dx dy will be easiest: hence

$$\mathbb{P}(X + Y \geq 2) = \int_0^1 \int_{2-y}^{\infty} 6 \cdot \frac{y^4}{x^4} \, \mathrm{d}x \, \mathrm{d}y$$

(c) Find $f_X(x)$, the marginal p.d.f. of X.

Solution: To find the marginal p.d.f. of X, we integrate out y. In order to do so, however, we must consider two cases:

- If $x \in [0, 1]$ then

$$f_X(x) = \int_0^x 6 \cdot \frac{y^4}{x^4} \, \mathrm{d}y = \frac{6}{x^4} \int_0^x y^4 \, \mathrm{d}x = \frac{6}{5} \cdot \frac{1}{x^4} \cdot x^5 = \frac{6}{5} x$$

- If $x \in [1, \infty]$ then

$$f_X(x) = \int_0^1 6 \cdot \frac{y^4}{x^4} \, \mathrm{d}y = \frac{6}{x^4} \int_0^1 y^4 \, \mathrm{d}x = \frac{6}{5x^4}$$

So, putting everything together,

$$f_X(x) = \begin{cases} \frac{6}{5} x & \text{if } x \in [0, 1] \\ \frac{6}{5x^4} & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

One can verify that this integrates to unity, as epxected.

(d) Find $f_Y(y)$, the marginal p.d.f. of Y.

Solution: For $f_Y(y)$, we do not need to split into cases: for $y \in [0, 1]$ we have

$$f_Y(y) = \int_y^{\infty} 6 \cdot \frac{y^4}{x^4} \, \mathrm{d}x = 6y^4 \cdot \left[-\frac{1}{3x^3} \right]_{x=y}^{x=\infty} = 2y^4 \cdot \frac{1}{y^3} = 2y$$

That is,

$$f_Y(y) = \begin{cases} 2y & \text{if } y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

(e) Find $\mathbb{E}[X]$

Solution: We could use $f_X(x)$ to compute the necessary integral, but I shall go ahead and apply the multidimensional version of the LOTUS to compute $\mathbb{E}[X]$ using a double integral:

$$\mathbb{E}[X] = \int_0^1 \int_y^{\infty} x \cdot 6 \cdot \frac{y^4}{x^4} \, \mathrm{d}x \, \mathrm{d}y$$

$$= 6 \int_0^1 \frac{y^4}{x^3} \, dx \, dy$$
$$= 6 \int_0^1 y^4 \cdot \left[-\frac{1}{2x^2} \right]_{x=y}^{x=\infty} dy$$
$$= 3 \int_0^1 y^4 \cdot \frac{1}{y^2} \, dy = 3 \int_0^1 y^2 \, dy = 1$$

One can verify that this is indeed the same answer we would get if we had computed $\mathbb{E}[X]$ using $\int_{-\infty}^{\infty} x f_X(x) \, dx$ instead.

(f) Find $\mathbb{E}[Y]$

Solution: We could again apply the LOTUS, but in this case it will be simpler to just use $f_Y(y)$:

$$\mathbb{E}[Y] = \int_0^1 2y^2 \, dy = \frac{2}{3}$$

(g) Compute $\text{Cov}(X, Y)$.

Solution: Recall that $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. The only term we have not yet computed is $\mathbb{E}[XY]$; to do so, we apply the multidimensional version of the LOTUS:

$$\mathbb{E}[XY] = \int_0^1 \int_y^\infty xy \cdot 6 \cdot \frac{y^4}{x^4} \, dx \, dy$$
$$= 6 \int_0^1 \frac{y^5}{x^3} \, dx \, dy$$
$$= 6 \int_0^1 y^5 \cdot \left[-\frac{1}{2x^2} \right]_{x=y}^{x=\infty} dy$$
$$= 3 \int_0^1 y^5 \cdot \frac{1}{y^2} \, dy = 3 \int_0^1 y^3 \, dy = \frac{3}{4}$$

Therefore,

$$\text{Cov}(X, Y) = \frac{3}{4} - (1) \left(\frac{2}{3} \right) = \frac{1}{12}$$

(h) Are X and Y independent? Explain.

Solution: There are several explanations for why X and Y are not independent. The first is to use the definition of independence; clearly $f_X(x) \cdot f_Y(y) \neq f_{X,Y}(x, y)$. Alternatively, we could have argued that if they *were* independent then they would be uncorrelated; since we have a nonzero correlation, we can conclude that we fail to have independence.

2. Let $X_i \sim \text{Exp}(\lambda_i)$ for $i = 1, \cdots, n$; further suppose that the X_i's are independndt. Define

$$Y := \min_{1 \leq i \leq n} \{X_i\}$$

in other words, Y is the smallest of the X_i's. Identify the distribution of Y **by name**; be sure to include any/all relevant parameter(s)!

Solution: Let's examine the survival function (i.e. one-minus-the-cdf) of Y:

$$\overline{F_Y}(y) := \mathbb{P}(Y \geq y) = \mathbb{P}\left(\min_{1 \leq i \leq n} \{X_i\} \geq y\right)$$

Now, note the following: if the smallest of n numbers is larger than y, then all n of those numbers must be larger than y. In other words,

$$\{Y \geq y\} = \bigcap_{i=1}^{n} \{X_i \geq y\}$$

and so

$$\overline{F_Y}(y) = \mathbb{P}\left(\bigcap_{i=1}^{n} \{X_i \geq y\}\right)$$

Since the X_i's are independent by assumption, we further have

$$\overline{F_Y}(y) = \mathbb{P}\left(\bigcap_{i=1}^{n} \{X_i \geq y\}\right) = \prod_{i=1}^{n} \mathbb{P}(X_i \geq y) = \prod_{i=1}^{n} (e^{-\lambda_i y}) = \exp\left\{-y \sum_{i=1}^{n} \lambda_i\right\}$$

To make things even more explicit, we can take the negative-derivative and see that

$$f_Y(y) = \left(\sum_{i=1}^{n} \lambda_i\right) e^{-(\sum_{i=1}^{n} \lambda_i) y}$$

meaning

$$Y \sim \mathrm{Exp}\left(\sum_{i=1}^{n} \lambda_i\right)$$

3. Let (X, Y) be a random vector with joint p.m.f. given by

$$p_{X,Y}(x, y) = \begin{cases} (y-1)\left(\frac{1}{2}\right)^{x+y} & \text{if } x \in \{1, 2, \cdots\},\ y \in \{2, 3, \cdots\} \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that $p_{X,Y}(x, y)$ is a valid joint p.m.f.

Solution: Nonnegativity is relatively trivial; $(1/2)^{x+y}$ will be nonnegative whenever $x \geq 1$ and $y \geq 2$, and the same is true of $(y-1)$. Thus, all that remains is to check that the p.m.f. sums to unity:

$$\begin{aligned} \sum_x \sum_y p_{X,Y}(x, y) &= \sum_{x=1}^{\infty} \sum_{y=2}^{\infty} (y-1)\left(\frac{1}{2}\right)^{x+y} \\ &= \left[\sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x\right] \cdot \left[\sum_{y=2}^{\infty} (y-1)\left(\frac{1}{2}\right)^y\right] \end{aligned}$$

$$= \frac{1/2}{1 - 1/2} \times \sum_{n=1}^{\infty} n \left(\frac{1}{2} \right)^{n+1}$$
$$= 1 \times \frac{1}{2} \sum_{n=0}^{\infty} n \left(\frac{1}{2} \right)^{n}$$
$$= \frac{1}{2} \cdot \frac{1/2}{(1 - 1/2)^2} = \frac{1/4}{1/4} = 1 \checkmark$$

Therefore, $p_{X,Y}(x, y)$ is a valid p.m.f..

(b) Find the marginal p.m.f.'s of X and Y. **Hint:** You can answer this question without doing any additional summations!

Solution: We could find the marginal p.m.f.'s using summation. However, note that we can factorize the joint:

$$(y-1)\left(\frac{1}{2}\right)^{x+y} = \left(\frac{1}{2}\right)^{x} \times (y-1)\left(\frac{1}{2}\right)^{y}$$
$$= \left(\frac{1}{2}\right)^{x-1}\left(\frac{1}{2}\right) \times \binom{y-1}{2-1}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{y-2}$$

This allows us to immediately note that $X \perp Y$; furthermore, we recognize the $x-$ and $y-$terms above, which allows us to conclude $X \sim \text{Geom}(1/2)$ and $Y \sim \text{NegBin}(2, 1/2)$.

(c) Use your answer to part (b) to compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$. **Hint:** You can answer this question without doing any additional summations!

Solution: Since $X \sim \text{Geom}(1/2)$ we have

$$\mathbb{E}[X] = \frac{1}{1/2} = 2$$

Similarly, since $Y \sim \text{NegBin}(2, 1/2)$ we have

$$\mathbb{E}[Y] = \frac{2}{1/2} = 4$$

(d) Compute $\mathbb{E}[XY]$. **Hint:** You can answer this question without doing any additional summations! Just be sure to justify all of your work/steps.

Solution: In part (b), we saw that $X \perp Y$ which means $\text{Cov}(X, Y) = 0$, and consequently $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y] = 2 \cdot 4 = 8$

4. Clara and Donna both roll a fair $k-$sided die, independently of each other.

a) Compute the probability that Clara rolls a number smaller than Donna.

Solution: Let C denote the value of Clara's roll and D denote the value of Donna's roll. Then,

$$C, D \overset{\text{i.i.d.}}{\sim} \text{Unif}\{1, 2, \dots, k\}$$

We seek $\mathbb{P}(C < D)$. Note that we can decompose the event $\{C < D\}$ in the following way:

$$\{C < D\} = \bigcup_{j=1}^{k} [\{C = j\} \cap \{D > j\}]$$

Since C and D are independent, and since all events in the union are disjoint, taking the probability of both sides of the above equation yields

$$\mathbb{P}(C < D) = \sum_{j=1}^{k} \mathbb{P}(C = j) \cdot \mathbb{P}(D > j)$$

From the information provided about the distribution of C and D, we find

$$\mathbb{P}(C = j) = \frac{1}{k}$$
$$\mathbb{P}(D > j) = 1 - \mathbb{P}(D \leq j) = 1 - \frac{j}{k}$$

Therefore:

$$\begin{aligned}
\mathbb{P}(C < D) &= \sum_{j=1}^{k} \mathbb{P}(C = j) \cdot \mathbb{P}(D > j) \\
&= \sum_{j=1}^{k} \frac{1}{k} \cdot \left(1 - \frac{j}{k}\right) \\
&= \frac{1}{k} \left[k - \frac{1}{\cancel{k}} \cdot \frac{\cancel{k}(k+1)}{2} \right] \\
&= 1 - \frac{k+1}{2k} = 1 - \frac{1}{2} - \frac{1}{2k} = \boxed{\frac{1}{2} - \frac{1}{2k}}
\end{aligned}$$

b) Consider the probability that Clara rolls a number strictly greater than Donna. A student argues that this probability should simply be 1 minus the answer to part (a). Do you agree with this student's reasoning? Explain why or why not.

Solution: The student's logic is flawed. This is because $\{C < D\}^{\complement} = \{C \geq D\}$; the student has neglected to take into account the fact that Clara and Donna could plausibly roll the same number.