HOMEWORK 7

PSTAT 120A: Summer 2022

Due: 11:00am on Thursday, July 21
Instructor: Ethan P. Marzban

Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.
- Be sure to show your work; correct answers with no supporting work will not be awarded full points.
- 2 randomly selected questions/parts will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

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1. A few lectures ago, we encountered the χ^2 distribution. In this problem, we shall investigate this distribution further.
   (a) If $T \sim \chi^2$, write down $f_T(t)$, the p.d.f. of T. (Yes, we did prove this in lecture! You don't need to re-derive the result; just write it down.)
   (b) Identify the χ^2 distribution as a special case of one of our familiar distributions.
   (c) Suppose $T_i \overset{\text{i.i.d.}}{\sim} \chi^2$ for $i = 1, \cdots, n$. Identify the distribution of $W := \sum_{i=1}^{n} X_i$. **Hint:** One of the results from Discussion Worksheet 7 might be useful here. **By the Way:** The distribution of W is called the **χ^2 distribution with n degrees of freedom**, and is notated $W \sim \chi^2_n$.
   (d) Now, suppose $X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ for $i = 1, \cdots, n$. Define

$$S := \frac{1}{n} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2$$

   Compute $\mathbb{E}[S]$ and $\text{Var}(S)$.

2. Recall the notion of **runs**, discussed in lecture: a run, in the context of tossing a $p-$coin n times, refers to a string of consecutive heads or tails. For instance, in the outcome

$$H\,H\,H\,T\,T\,H\,T\,H\,T\,T$$

   there are 6 runs:

$$H\,H\,H\ |\,T\,T\,|\ H\,|\ T\,|\ H\,|\ T\,T$$

   Let X denote the number of runs in n tosses of a $p-$coin. We will also make the simplifying assumption that n is even.
   (a) What is the state space of X?
   (b) Compute $\mathbb{P}(X = 1)$.
   (c) Compute $\mathbb{P}(X = n)$.

   It turns out that the PMF doesn't have a nice simple expression. Nonetheless, we can actually find a relatively simple expression for $\mathbb{P}(X = 2)$, which is what we will work toward in the next few parts.

   (d) Suppose the first toss lands heads. Compute the probability of exactly 2 runs in this case.
   (e) Now, suppose the first toss lands tails. Compute the probability of exactly 2 runs in this case.
   (f) Combine your two cases above to conclude

$$\mathbb{P}(X = 2) = 2 \cdot \frac{p^n q - p q^n}{p - q}$$

3. Given a collection of random variables $\{X_i\}_{i=1}^{n}$, we define the **sample mean** to be

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$$

For the purposes of this problem, we will assume that the X_i's are i.i.d. with common mean μ and common variance σ^2; i.e. $\mathbb{E}[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$ for all $i = 1, \cdots, n$, and the X_i's are all independent.

(a) Compute $\mathbb{E}[\bar{X}_n]$ as a function of μ and n.

(b) Compute $\text{Var}(\bar{X}_n)$ as a function of σ^2 and n.

(c) If $X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, what is the distribution of $\bar{X}_n$? Confirm that your answers to parts (a) and (b) are consistent with your answer to this part.

(d) If $X_i \overset{\text{i.i.d.}}{\sim} \text{Gamma}(r, \lambda)$, what is the distribution of $\bar{X}_n$? Confirm that your answers to parts (a) and (b) are consistent with your answer to this part.

4. The following parts are unrelated.

(a) Let X be a continuous random variable with MGF given by

$$M_X(t) = \begin{cases} (1-t)^{-3/2} & \text{if } t < 1 \\ \infty & \text{otherwise} \end{cases}$$

Find $\mathbb{E}[X]$ and $\text{Var}(X)$. **DO NOT** simply recognize the distribution of X; this part is designed to give you practice with differentiation!

(b) Let X be a discrete random variable with MGF (Moment-Generating Function) given by

$$M_X(t) = (0.3 + 0.7e^{2t})^8$$

Find $p_X(k)$, the probability mass function (p.m.f.) of X.