

## EXAMPLES FROM SLIDE DECK 14

PSTAT 120A: Summer 2022

Instructor: Ethan P. Marzban

1. Suppose that I am a shopowner, and I know that the number of customers arriving at my shop follows a Poisson distribution. However, I also know that the rate at which customers arrive at my store is much lower on rainy days as opposed to dry days. Specifically, on rainy days customers arrive at a rate  $\lambda_r$  per day, and on dry days customers arrive at a rate  $\lambda_d$  per day. Further suppose that there is a  $p = 10\%$  chance of rain tomorrow.

(a) If  $X$  denotes the number of customers that will arrive at my store tomorrow, what is the p.m.f. (probability mass function) of  $X$ ?

**Solution:** Let  $R$  denote the event "it rains tomorrow." Then, we have

$$(X | R) \sim \text{Pois}(\lambda_r)$$

$$(X | R^c) \sim \text{Pois}(\lambda_d)$$

$$\mathbb{P}(R) = 0.1$$

Therefore,

$$\begin{aligned} p_X(k) &= \mathbb{P}(X = k | R)\mathbb{P}(R) + \mathbb{P}(X = k | R^c)\mathbb{P}(R^c) \\ &= e^{-\lambda_r} \cdot \frac{\lambda_r^k}{k!} \cdot 0.1 + e^{-\lambda_d} \cdot \frac{\lambda_d^k}{k!} \cdot 0.9 \end{aligned}$$

(b) What  $\mathbb{E}[X]$ ?

**Solution:**

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X | R]\mathbb{P}(R) + \mathbb{E}[X | R^c]\mathbb{P}(R^c) \\ &= 0.1\lambda_r + 0.9\lambda_d \end{aligned}$$

2. Suppose  $(X, Y)$  is a continuous bivariate random vector with joint p.d.f. given by

$$f_{X,Y}(x,y) = \begin{cases} \lambda^3 x e^{-\lambda y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) Find  $f_Y(y)$ , the marginal density of  $Y$ . Use this to compute  $\mathbb{E}[Y]$ .

**Solution:**

$$\begin{aligned}
 f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \\
 &= \int_0^y \lambda^3 x e^{-\lambda y} \, dx = \lambda^3 \cdot \frac{1}{2} y^2 e^{-\lambda y} \cdot \mathbb{1}_{\{y \geq 0\}}
 \end{aligned}$$

We recognize that  $Y \sim \text{Gamma}(3, \lambda)$ , meaning  $\mathbb{E}[Y] = 3/\lambda$ .

(b) Find  $f_{X|Y}(x | y)$ , the conditional density of  $(X | Y = y)$ **Solution:**

$$\begin{aligned}
 f_{X|Y}(x | y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
 &= \frac{\lambda^3 x e^{-\lambda y} \cdot \mathbb{1}_{\{0 \leq x \leq y\}} \cdot \mathbb{1}_{\{y \geq 0\}}}{\lambda^3 \cdot \frac{1}{2} y^2 e^{-\lambda y} \cdot \mathbb{1}_{\{y \geq 0\}}} = \frac{2x}{y^2} \cdot \mathbb{1}_{\{0 \leq x \leq y\}}
 \end{aligned}$$

(c) Compute  $\mathbb{P}(X \geq 1 | Y = 2)$ .**Solution:**

$$\begin{aligned}
 \mathbb{P}(X \geq 1 | Y = 2) &= \int_1^{\infty} f_{X|Y}(x | 2) \, dx \\
 &= \int_1^2 \frac{2x}{2^2} \cdot \mathbb{1}_{\{0 \leq x \leq 2\}} \, dx = \frac{1}{4} \int_1^2 2x \, dx = \frac{3}{4}
 \end{aligned}$$

(d) Compute  $\mathbb{E}[X | Y = 1]$ **Solution:**

$$\begin{aligned}
 \mathbb{E}[X | Y = 1] &= \int_{-\infty}^{\infty} x f_{X|Y}(x | 1) \, dx \\
 &= \int_0^1 x \cdot \frac{2x}{1^2} \cdot \mathbb{1}_{\{0 \leq x \leq 1\}} \, dx = \frac{2}{3}
 \end{aligned}$$