1: Foundations of Probability, and Counting

PSTAT 120A: Summer 2022

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Welcome!

- Welcome to PSTAT 120A!
- About me:

Name:	Ethan
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OH:	T, Th 4 - 5:30pm (PDT)
OH Location:	South Hall 5421 (the "StatLab")

- Important Course Resources:
  - Gauchospace (main course website)
  - Public Course Website: pstat120a.github.io
  - Gradescope: for submitting HW (and receiving feedback), and quizzes

Logistics

- **Homeworks: TWO** sets per week; one due Tuesday, one due Friday.
  - Admittedly, perhaps a bit more challenging than the lecture examples... but not too much more!
  - Collaboration is encouraged; just please ensure that whatever you submit is your own work.

- **Quizzes:** 3-5 questions, 20 minutes, **remote (asynchronous) on Thursdays.** Take place on Gradescope, between 6pm and 11:59pm.
  - Not designed to be overly challenging; if you're keeping up with lecture material and homework, I don't think you should have any trouble!

- **Exams:** One midterm and one Final
  - Midterm: Thursday, July 7 from 2 - 3:05pm (during lecture)
  - Final: Thursday July 28, **4 - 7pm**

- Please read the syllabus!

Probability

Smithsonian Magazine

Why It's So Hard to Make Risk Decisions in the Pandemic

Humans take dozens of cognitive shortcuts. There's optimism bias. When asked to rate their own chances of getting Covid instead of their peers,...

1 day ago

The Mary Sue

Who will be Doctor Who's 14th Doctor?

Doctor Who, following the adventures of an errant Time Lord and their series of companions, has been airing for a whopping 60 years (albeit with...

Feb 22, 2022

CBS Sports

2022 AT&T Byron Nelson odds, field: Surprising PGA picks, predictions from golf model that's nailed 8 majors

2022 AT&T Byron Nelson odds, field: Surprising PGA picks, predictions from golf model that's nailed 8 majors. By CBS Sports Staff. 2 hrs ago • 4 min read.

3 hours ago

Probability

- Uncertainty surrounds us!
- Statistics is, in many ways, the study of uncertainty.
- Probability is the **language** of uncertainty; it gives us a way to place our beliefs in an uncertain world against a rigorous mathematical backdrop.
    - Yes- **math!**
    - Probability is, in its truest form, an offshoot of mathematics.
    - Hence, there is a need for a strong mathematical background.
- Please make sure you are *very* comfortable with calculus before coming into this course.
    - I've posted some review material on our course sites for you to consult, if you feel your calculus isn't quite up to par,
    - I also welcome any questions about this during office hours!

The Key Ingredients

- We start with the notion of a **experiment**.

Definition

An **experiment** is a procedure that can be repeated an infinite number of times, where each time we repeat the procedure there are a fixed set of things (called **outcomes**) that could occur.

- Some examples:
  - Tossing a coin twice
  - Rolling a die
  - Sampling from a population

The Key Ingredients

- What happens when we put all of the outcomes associated with a particular experiment into one giant set?

Definition

The **outcome space**, denoted with the capital greek letter Ω, is the set consisting of all outcomes associated with a particular experiment.

- We can also broaden our horizons, and consider things that are slightly more complicated than single outcomes:

Definition

An **event** is a subset of the outcome space. The set of all events associated with a particular outcome space (i.e. a set containing subsets of Ω) is called the **event space**, and is denoted by $\mathcal{F}$.

  - We won't talk too much about event spaces in this class, but they are an important ingredient in the concept we have been working our way toward...

Wait a Minute–

- Before we go any further, notice how there's been a lot of mention about **sets**.

- I know sets aren't always covered in great detail in calculus, but they are a crucial ingredient in probability (as we are starting to see!)
- As such, I'd like to take a quick detour to talk about some basics of Set Theory.

Crash Course on Set Theory

Sets

- A **set** is an unordered collection of objects, called **elements**.
  - The elements of a set needn't be numbers; $\{\text{red}, x, ☺\}$ is a perfectly valid set.
  - What do we mean by "unordered?" It means $\{1, 2, 3\}$ and $\{3, 2, 1\}$ are the same set, because they contain precisely the same elements.
  - To notate the fact that some quantity x is an element of a particular set S, we will use the notation $x \in S$.

- How can we "compare" sets?
  - One idea is to look at the *number of elements* in a set: we define, loosely speaking, the **cardinality** of a set A (notated $|A|$, or $\#(A)$) to be the number of elements in a set.
  - The notion of cardinality becomes a bit trickier when there are an infinite number of elements in our set... We'll cross that bridge when we come to it, though!

  - Ok, what about this situation: $A = \{1, 2, 3, 4\}$ and $B = \{2, 3\}$. Clearly B is "inside" A! This brings us to the notion of *subsets*: we say a set B is a **subset** of a set A (notated $B \subseteq A$) if every element of B is also an element of A.

Sets

- We have a few so-called **set operations:**
  - **Union:** $A \cup B := \{x : x \in A \text{ or } x \in B\}$
  - **Intersection:** $A \cap B := \{x : x \in A \text{ and } x \in B\}$. Sometimes notated AB.
  - **Complement:** $A^{\complement} := \{x : x \notin A\}$
  - **Set Difference:** $A \setminus B := \{x : x \in A \text{ and } x \notin B\}$

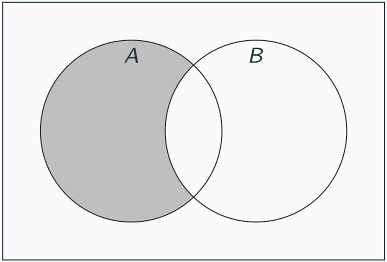
- A note on complements: strictly speaking, complements can only be defined in the context of some larger set to which A belongs, called the **universe of discourse**. Thankfully, in the context of probability, we will almost always be working inside of the outcome space, so this point won't really be an issue.

- One other special quantity is that of the **empty set:** the empty set (notated $\varnothing$) is the set containing no elements.

- As a concrete illustration of some of these set operations: let $A = \{1, 2, 3, 4\}$ and $B = \{4, 5\}$. Then:
  - $A \cup B = \{1, 2, 3, 4, 5\}$
  - $A \cap B = \{4\}$

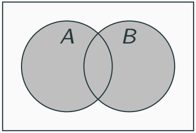
Venn Diagrams

- A very useful tool in visualizing sets and set relation is that of the **Venn Diagram**.

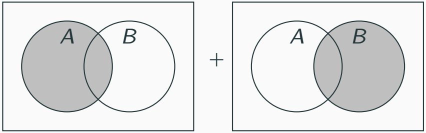


- What set is this?

- Venn Diagrams are especially useful in deriving new relationships between sets!
- For example, how might we compute $|A \cup B|$?
- Here is the set we're interested in:



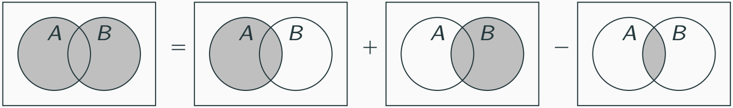
- Naïvely, we may say $|A \cup B| = |A| + |B|$:



- But, we've overcounted! By how much? $|A \cap B|$.

The Addition Rule

- Pictorially,



- Mathematically,

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1}$$

which is sometimes known as the **Addition Rule**.

  - Let me stress: subtracting off $|A \cap B|$ does **NOT** mean that $A \cup B$ translates to "either A or B but not both"; that is something known as an **exclusive or** (or **symmetric difference**). Rather, subtracting $|A \cap B|$ is only a means of accounting for our overcounting!

- This generalizes to something called the **Inclusion-Exclusion Rule**, which we will discuss later.

Back to Probability

Probability

- Wait- I thought we were in a *Probability* class, not a Set Theory Class!
- Ok ok, fair enough... let's talk about Probability!
- Colloquially, **Probability** represents our beliefs on something; specifically, on a particular event.
  - E.g. "chance of rain; odds of winning big at a Casino, etc."
- Mathematically, what this means is that "Probability" takes an event, and spits out a percent (or fraction).
- Sounds like a function!

Probability Measures

Definition: Probability Measure

A **probability measure** associated with a particular experiment with outcome space Ω and event space $\mathcal{F}$ is a function

$$\mathbb{P} : \mathcal{F} \to \mathbb{R}$$

that satisfies three conditions:

1. $(\forall A \in \mathcal{F})[\mathbb{P}(A) \geq 0]$
2. $\mathbb{P}(\Omega) = 1$
3. For a sequence of pairwise disjoint events $A_1, A_2, \cdots$, [i.e. $A_i \cap A_j = \varnothing$ for any $i \neq j$], we have

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

Probability Measures

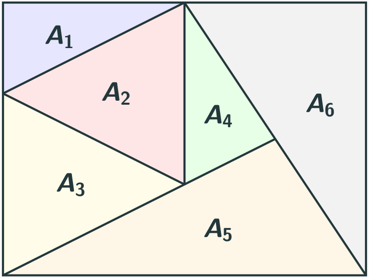
- Conditions (1) through (3) on the previous page are often referred to as the **Axioms of Probability.** The third Axiom is sometimes called the **countable additivity** of probability.
- Let's unpack these a bit.
  - (1) says that we want the probability of any event to be nonnegative; i.e. we are restricting ourselves from saying things like "the chance of rain is -25%".
  - (2) says that the probability of *something* happening is 100%.
  - (3) relates to something known as a **partition**.

Partitions

Definition: Partition

A sequence of sets $\{A_1, A_2, \cdots\}$ is said to **partition** (or **form a partition of**) a larger set A if

1. $\{A_i\}_{i=1}^{\infty}$ are pairwise disjoint [i.e. $A_i \cap A_j = \varnothing$ for any $i \neq j$]
2. $A = \bigcup_{i=1}^{\infty} A_i$



Partitions

- Note that, given an event B that is a subset of a particular outcome space Ω, $\{B, B^{\complement}\}$ forms a partition of Ω.

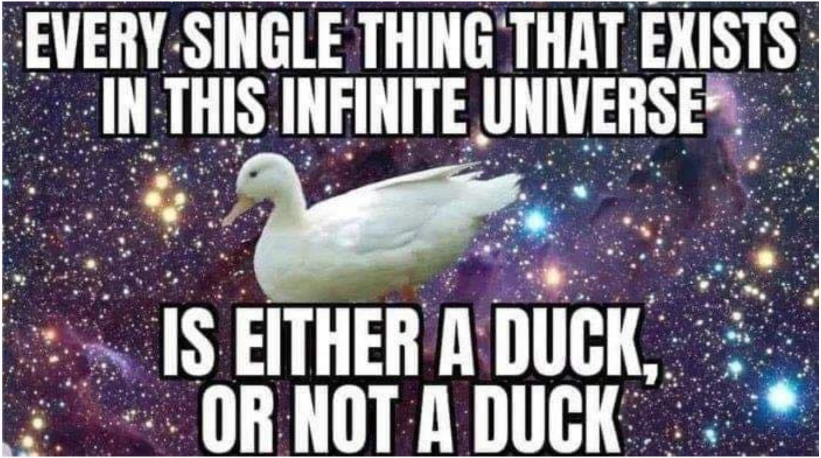
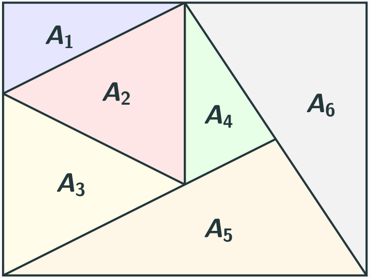


Figure 1: Source:

Countable Additivity

- Back to the countable additivity of probability.



- What the third axiom says is this: if we sum up the probabilities of the A_i's, we should get the probability of their union.

- Seems intuitive enough, right?

Probability Space

Definition: Probability Space

Consider an experiment with:

- Outcome Space Ω
- Event Space $\mathcal{F}$
- Probability Measure $\mathbb{P}$

Then the object $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **Probability Space**.

- Note that a probability space is just a collection of three objects: two collections of sets, and one function!
- If you go onto more advanced probability, you'll talk more about the mechanics of Probability Spaces. But, for the purposes of this class, you don't need to know much about their intricacies!

Theorem: Complement Rule

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an event $A \in \mathcal{F}$, we have

$$\mathbb{P}(A^{\complement}) = 1 - \mathbb{P}(A)$$

Proof.

- $\{A, A^{\complement}\}$ forms a partition of Ω.
- Therefore, by the Third Axiom of Probability,

$$\mathbb{P}(A \cup A^{\complement}) = \mathbb{P}(A) + \mathbb{P}(A^{\complement})$$

- But, $A \cup A^{\complement} = \Omega$, and by the Second Axiom of Probability, $\mathbb{P}(\Omega) = 1$.
- Hence, putting these points together, we find

$$\mathbb{P}(A \cup A^{\complement}) = \mathbb{P}(\Omega) = 1 = \mathbb{P}(A) + \mathbb{P}(A^{\complement})$$

- Rearranging terms yields the desired result.

□

Deriving Probabilistic Identities

Theorem: Set Difference Rule

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an event $A \in \mathcal{F}$, we have

$$\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$$

Proof.

- $\{(A \setminus B), (A \cap B)\}$ forms a partition of A. (**Sketch a Picture!**)
- Therefore, by the Third Axiom of Probability,

$$\mathbb{P}([A \setminus B] \cup [A \cap B]) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$$

- But, $[A \setminus B] \cup [A \cap B] = A$.
- Hence, putting these points together, we find

$$\mathbb{P}([A \setminus B] \cup [A \cap B]) = \mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$$

- Rearranging terms yields the desired result.

□

Deriving Probabilistic Identities

Theorem: The Addition Rule

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two events $A, B \in \mathcal{F}$, we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Proof.

- $\{(A \setminus B), (A \cap B), (B \setminus A)\}$ forms a partition of $(A \cup B)$.
- Therefore, by the Third Axiom of Probability,

$$\mathbb{P}([A \setminus B] \cup [A \cap B] \cup [B \setminus A]) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)$$

- But, $[A \setminus B] \cup [A \cap B] \cup [B \setminus A] = (A \cup B)$.
- Hence, putting these points together, we find

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)\\ &= [\mathbb{P}(A) - \cancel{\mathbb{P}(A \cap B)}] + \cancel{\mathbb{P}(A \cap B)} + [\mathbb{P}(B) - \mathbb{P}(A \cap B)]\\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\end{aligned}$$

Equally Likely Outcomes

- So far, we've been dealing only with abstract probability measures. What are some concrete examples of probability measures?
- Here's one, that goes by many names: we'll call it the **Classical Definition of Probability**, but sometimes it is called the **equally likely probability measure**:

Definition: Classical Definition of Probability

Consider an outcome space Ω with n elements. Then, the probability measure defined by

$$\mathbb{P}(A) = \frac{|A|}{n}$$

is a valid probability measure.

- Why the name? This is the probability measures that arises from experiments in which outcomes are all equally likely.
  - For instance: rolling a **fair** die, flipping a **fair** coin, drawing a number **at random** from a set of numbers, etc.
- You can verify that this satisfies the three axioms of probability.

Equally Likely Outcomes

Example 1: Suppose I toss a fair coin once.

- Let H denote "the coin landed heads" and T denote "the coin landed tails."
- $\Omega = \{H, T\}$
- Because the coin is fair, we can assume equally likely outcomes. That is,

$$\mathbb{P}(H) = \frac{1}{2}; \quad \mathbb{P}(T) = \frac{1}{2}$$

- Note that we could have written $H^{\complement}$ in place of T.
- Kind of boring...

Equally Likely Outcomes

Example 2: Suppose I toss a fair coin twice.

- Let H denote "the coin landed heads" and T denote "the coin landed tails."
- $\Omega = \{H, T\}^2 = \{(x, y) : x \in \{H, T\},\ y \in \{H, T\}\}$
- Because the coin is fair, we can assume equally likely outcomes. That is,

$$\mathbb{P}((x, y)) = \frac{1}{4} \quad \forall (x, y) \in \Omega$$

- Let A denote the event "I observed at least one heads."
- $A = \{(H, H), (H, T), (T, H)\}$
- So $\mathbb{P}(A) = \frac{|\{(H,H)(H,T),(T,H)\}|}{4} = \boxed{\frac{3}{4}}$

Equally Likely Outcomes

- Easier way to get $\mathbb{P}(A)$? Yes!
- Note that $A^{\complement}$ means "I observed **no** heads", so

$$A^{\complement} = \{(T, T)\}$$

and

$$\mathbb{P}(A^{\complement}) = \frac{|\{(T, T)\}|}{4} = \frac{1}{4}$$

meaning, by the complement rule,

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^{\complement}) = 1 - \frac{1}{4} = \frac{3}{4} \checkmark$$

- **Moral:** Always check to see if the probability of the complement is easier to compute!

Equally Likely Outcomes

- In addition to our event A from before, now let B denote the event "I observed at least one Tails." Suppose we wish to compute $\mathbb{P}(A \cup B)$.
- We immediately see

$$\mathbb{P}(A) = \frac{3}{4}$$
$$\mathbb{P}(B) = \frac{|\{(H,T),(T,H),(T,T)\}|}{4} = \frac{3}{4}$$

- Additionally, $A \cap B$ denotes "I observed at least one heads and at least one tails".
- Mathematically,

$$A \cap B = \{(H,T),(T,H)\}$$

and so

$$\mathbb{P}(A \cap B) = \frac{|\{(H,T),(T,H)\}|}{4} = \frac{1}{2}$$

- Therefore, by the Addition Rule,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{3}{4} + \frac{3}{4} - \frac{1}{2} = 1$$

Equally Likely Outcomes

- Easier way? Yup!
- Note $(A \cup B)^{\complement}$ means "no heads and no tails." But this is impossible! Therefore, $\mathbb{P}([A \cup B]^{\complement}) = 0$, and so $\mathbb{P}(A \cup B) = 1 - 0 = 1$.

- In fact, there are two set-theory formulas I didn't mention before that could be useful:

Theorem: DeMorgan's Laws

Given two sets A and B, we have

- $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$
- $(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$

More generally, for a sequence of events $\{A_i\}_{i=1}^{\infty}$, we have

- $\left(\bigcup_{i=1}^{\infty} A_i \right)^{\complement} = \bigcap_{i=1}^{\infty} A_i^{\complement}$
- $\left(\bigcap_{i=1}^{\infty} A_i \right)^{\complement} = \bigcup_{i=1}^{\infty} A_i^{\complement}$

A Note on Outcome Spaces

- Could I have written the outcome space as $\{0, 1\}^2$? Sure!?
  - OK, in words, what does the outcome $(0, 1)$ mean?
  - That's right- we don't really know!
  - Because we were not **clear** and **explicit** about our notation.
  - If I say "let 0 denote 'tails' and 1 denote 'heads' ", then the above is a perfectly valid outcome space. (Same if we reversed the roles of 0 and 1)

- **Moral of the Story:** outcome spaces are not unique; they are inherently linked with our notation. As such, it is important to be **clear** and **explicit** about our notation.

Try it Yourself!

Example 3: Suppose I roll two fair six-sided dice.

(a) Write down a potential outcome space associated with this experiment. (Be explicit about your notation!)

(b) Can we use the equally likely probability measure here?

(c) Let A denote the event "the maximum of the two numbers is 3" and B denote "the sum of the two rolls was 4". Compute $\mathbb{P}(A \cup B)$

Leadup to our Next Topic...

- The Equally Likely probability measure is very useful!
- But, note that the probability of an event really only depend on the *number* of elements in that event.
- Up until now, we've been finding that number by explicitly listing out the elements in our event and then counting them up.
- Wouldn't it be nice if there is a way to systematically count the elements in a set, without having to list out all of the elements?

Counting/Combinatorics

Counting

- The field of **counting** (and **combinatorics**) is primarily concerned with systematically counting the elements in a set/event, without having to explicitly enumerate all of the elements in that set.

Fundamental Principle of Counting

Theorem: Fundamental Principle of Counting

If an experiment consists of k independent stages, and the i^{th} stage has a total of n_i configurations, then the total number of configurations in the experiment is

$$\prod_{i=1}^{k} n_i := n_1 \times n_2 \times \cdots \times n_k$$

- **Simple Example:** Suppose that at a particular ice cream shop, a "scoop" consists of one flavor and one topping. If there are 32 flavors and 8 toppings available, how many scoops can be created?
  - $32 \times 8 = 256$.

The Slot Method

- When utilizing the Fundamental Principle of Counting (FPC), a useful diagrammatic tool is that of the **slot method**. Here's how it works:
  1. Write down k blank lines ("slots"), where k is the number of stages in the experiment.
  2. In the i^{th} slot, write down n_i, the number of configurations corresponding to that stage.
  3. Finally, multiply across the slots to find the total number of configurations.

- Let's apply this to the ice cream example:

$$\underline{\ 32\ } \times \underline{\ \ 8\ \ }$$

- What if we say that a scoop also consists of a drizzle, and there are 4 drizzles available?

$$\underline{\ 32\ } \times \underline{\ \ 8\ \ } \times \underline{\ \ 4\ \ }$$

Reorderings

- Suppose I have n tickets, numbered 1 through n, in front of me. How many ways are there to arrange these n tickes in a line?
- Let's use the slot method. I will draw n slots, where each slot corresponds to a ticket:

$$\underline{\ n\ } \times \underline{\ n-1\ } \times \cdots \times \underline{\ 3\ } \times \underline{\ 2\ } \times \underline{\ 1\ }$$

- This type of quantity arises so often, we give it a name:

Definition: Factorial

For a natural number $n \in \mathbb{N}$, we define the quantity $n!$ (read, "n **factorial**") to be

$$n \times (n-1) \times \cdots \times 3 \times 2 \times 1$$

We definitionally set $0! = 1$.

- $3! = 3 \times 2 \times 1 = 6$
- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Picking Objects

- Now, let's consider a very simple example. Suppose I have a box with three tickets: one labelled A, one labelled B, and one labelled C.
- From this box, I will pick two tickets, **not** replacing the ticket I selected first.
- This is an experiment! So, it musth have an outcome space. My question is: **how many elements are in the outcome space?**

- Before we answer this question, we need a bit more information. Specifically, we need to know: **does order matter**?
- OK, what does it mean for "order to matter?"
- Think about a license plate: the plates `123ABC` and `312BCA` are clearly two different license plate, despite the fact that they are comprised of the same letters and numbers!

- In the context of this problem, asking whether or not order matters equates to asking ourselves "is picking A followed by B different than picking B followed by A?"

Order Matters

- Let's suppose order **does** matter.
- Then, letting (X, Y) denote the outcome "I drew the ticket labelled X first, then the ticket labelled Y second" (for $X \in \{A, B, C\}$ and $Y \in \{A, B, C\}$), then

$$\begin{aligned}\Omega = \{&(A, B),\ (A, C)\\ &(B, A),\ (B, C)\\ &(C, A),\ (C, B)\}\end{aligned}$$

  - Why didn't I include outcomes like (A, A)?
  - Because I'm not replacing my first ticket! Thus, if I drew the ticket A first, it is impossible for me to draw it again (because after I drew it, I didn't put it back into the box!)

- Therefore, the answer to "how many elements are in the outcome space" is 6.

Order Matters

- So, we answered the question by listing out all of the elements in Ω.
- But I thought the whole point of Counting was to enable us to answer this question *without* enumerating the elements in the outcome space!
- You're right- let's find a more systematic way to answer this question.
- Surprisingly, that systematic way is.... the slot method!

- Perhaps that's not so surprising. Our experiment consists of **2** stages (the two tickets we draw); thus, let's draw 2 slots:

$$\underline{\quad} \quad \underline{\quad}$$

- For the first stage, there are 3 possibilities (either ticket A, ticket B, or ticket C); thus we put a 3 in the first slot:

$$\underline{\;3\;} \quad \underline{\quad}$$

- Once we pick a ticket, there are now only $3 - 1 = 2$ tickets left in the box. Therefore, we put a 2 in our final slot:

$$\underline{\;3\;} \quad \underline{\;2\;}$$

- Multiply together (by the FPC) to find the answer is

$$\underline{\;3\;} \times \underline{\;2\;}$$

Order Matters

- Let's try a slightly more complicated example. Suppose now that I have 5 tickets, labeled A through E, and I now want to draw 3. How many ways are there to do this?
- We draw 3 slots, one for each of the tickets:

$$\_\_\_ \quad \_\_\_ \quad \_\_\_$$

- Now, there are 4 tickets in the box on our first draw, and $4 - 1 = 3$ tickets left on our second, and $3 - 1 = 2$ on our third. Thus:

$$\underline{\ 5\ } \times \underline{\ 4\ } \times \underline{\ 3\ }$$

so there are $5 \times 4 \times 3 = 60$ possible ways to select 3 tickets from a total of 4 without replacement, when order matters.

Order Matters

- Don't believe me?

$$\begin{aligned}\Omega = \{&(A,B,C),\ (A,B,D),\ (A,B,E),\ (A,C,B),\ (A,C,D),\ (A,C,E),\ (A,D,B),\ (A,D,C),\ (A,D,E),\ (A,E,B),\ (A,E,C),\ (A,E,D),\\ &(B,A,C),\ (B,A,D),\ (B,A,E),\ (B,C,A),\ (B,C,D),\ (B,C,E),\ (B,D,A),\ (B,D,C),\ (B,D,E),\ (B,E,A),\ (B,E,C),\ (B,E,D),\\ &(C,A,B),\ (C,A,D),\ (C,A,E),\ (C,B,A),\ (C,B,D),\ (C,B,E),\ (C,D,A),\ (C,D,B),\ (C,D,E),\ (C,E,A),\ (C,E,B),\ (C,E,D),\\ &(D,A,B),\ (D,A,C),\ (D,A,E),\ (D,B,A),\ (D,B,C),\ (D,B,E),\ (D,C,A),\ (D,C,B),\ (D,C,E),\ (D,E,A),\ (D,E,B),\ (D,E,C),\\ &(E,A,B),\ (E,A,C),\ (E,A,D),\ (E,B,A),\ (E,B,C),\ (E,B,D),\ (E,C,A),\ (E,C,B),\ (E,C,D),\ (E,D,A),\ (E,D,B),\ (E,D,C)\}\end{aligned}$$

Order Matters

- Finally, let's generalize this!
- Suppose we have n tickets (for some fixed $n \in \mathbb{N}$), and we wish to draw k (where $k \in \mathbb{N}$ and $k \leq n$) **without replacement** (i.e. we don't replace tickets in between draws). If order matters, how many possible configurations of $k-$tickets are possible?
- We use the slot method:

$$\underline{\quad n \quad} \times \underline{\quad n-1 \quad} \times \underline{\quad n-2 \quad} \times \cdots \times \underline{\quad n-k+1 \quad}$$

- We can actually write this more concisely using factorials:

$$n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

This quantity shows up so often, we give it a special notation and name: we call it ***n* order *k*** and denote it $(n)_k$:

$$(n)_k := \frac{n!}{(n-k)!} = n \times (n-1) \times (n-k+1)$$

  - Note that $(n)_n = n!$

Order Matters

- This is consistent with our work from above:
  - $(3)_2 = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3! = 6$
  - $(5)_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times \cancel{2} \times \cancel{1}}{\cancel{2} \times \cancel{1}} = 5 \times 4 \times 3 = 60$

Order Doesn't Matter

- Let's go back to our 3-ticket example: we have 3 tickets labeled A through C, and we want to draw 2 without replacement.
- When order matters, there are $(3)_2 = 6$ ways of doing this.
- What if ordered **doesn't** matter? That is, what if the outcomes (A, B) and (B, A) were not counted separately?

- From a enumerative standpoint,

$$\Omega = \{(A, B),\ (A, C),\ (B, C)\}$$

so $|\Omega| = 3$. But, again, we want to find a way to count *without* listing out the elements!

Order Doesn't Matter

- Here's one way to think about where this 3 came from:
  - Start by supposing order *did* matter; then there are 6 possibilities.
  - Now, divide through by the number of ways to reorder 2 elements among themselves. Why is this? Well, once we've picked the *letters* in our sample, all outcomes that contain those same letters but just shuffled around will be considered indistinguishable!

- Let's jump to the general case. If we have n tickets and want to sample k without replacement, where order **doesn't** matter, then the total number of ways to do this is

suppose order mattered

$$\frac{(n)_k}{k!}$$

group like terms

Order Doesn't Matter

- This quantity also arises frequently, so we give *it* a name as well: we call this ***n* choose *k*** and denote it $\binom{n}{k}$:

$$\binom{n}{k} := \frac{(n)_k}{k!} = \frac{\left(\frac{n!}{(n-k)!}\right)}{k!} = \frac{n!}{k! \cdot (n-k)!}$$

- To Summarize: the number of ways to sample k objects from n without replacement is:
  - $(n)_k := \dfrac{n}{(n-k)!}$ if order **does** matter
  - $\dbinom{n}{k} := \dfrac{n!}{k! \cdot (n-k)!}$ if order **doesn't** matter

Sampling With Replacement

- One final assumption we can examine further is that of sampling with or without replacement.
- In our example above, I did *not* replace tickets between each successive draw. What happens if I *do* replace tickets?
- Specifically, consider the following situation: I have n objects, and I want to select k of them with replacement.
- Unsurprisingly, we draw a slot diagram!

$$\underbrace{\underline{\quad n \quad} \times \underline{\quad n \quad} \times \underline{\quad n \quad} \times \cdots \times \underline{\quad n \quad}}_{k \text{ slots}}$$

- So, the number of ways to select k objects from a total of n with replacement is n^k.

- Let's put everything together by way of a few example.

License Plates

California state license plates consist of 7 characters: a digit, followed by 3 letters, followed by 3 digits.

- Suppose we do not allow letters or digits to be repeated; that is, plates like `A122BCC345` are not valid, whereas `A123BCD456` is valid. How many license plates can be made according to this labelling scheme?
- As mentioned before, the order in which digits appear in a license plate matters. Additionally, since we are assuming repeated digits/letters are not allowed, we are effectively sampling without replacement.
- The act of choosing a plate can be decomposed into 7 stages, where the first stage corresponds to picking the initial letter, the second corresponds to picking the first digit, and so on and so forth. Therefore, using a slot diagram, we place down 7 slots:

$$\underline{\quad 10 \quad} \times \underline{\quad 26 \quad} \times \underline{\quad 25 \quad} \times \underline{\quad 24 \quad} \times \underline{\quad 9 \quad} \times \underline{\quad 8 \quad} \times \underline{\quad 7 \quad}$$

- Therefore, the final answer is $10 \cdot (26)_3 \cdot (9)_3 = (10)_4 \cdot (26)_3$

License Plates

California state license plates consist of 7 characters: a digit, followed by 3 letters, followed by 3 digits.

- Admittedly, the license plate labelling scheme in reality doesn't prohibit repeated digits/letters; that is, a license plate like `A122BCC345` is perfectly valid. In this more realistic labelling scheme, how many license plates can be created?
- Again, order matters. Now, however, we are effectively sampling *with* replacement.
- Our slot diagram then looks like

$$\underline{\quad 10 \quad} \times \underline{\quad 26 \quad} \times \underline{\quad 26 \quad} \times \underline{\quad 26 \quad} \times \underline{\quad 10 \quad} \times \underline{\quad 10 \quad} \times \underline{\quad 10 \quad}$$

- Therefore, the final answer is $10 \cdot (26)^3 \cdot (10)^3 = (10)^4 \cdot (26)^3$

Basic Counting Principles:

Selecting k objects from a total of n:

	Order of Selection is Important	Order of Selection is Unimportant
Repetitions allowed (w/ replacement)	n^k	$\binom{n+k-1}{k-1}$
Repetitions **not** allowed (w/o replacement)	$(n)_k = \frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

One Final Fact

- One fact I forgot to mention explicitly (but one that we used implicitly already!) is the following:

Theorem

$$\left| \bigtimes_{i=1}^{n} A_i \right| = \prod_{i=1}^{n} |A_i|$$

Coin Toss: Chalkboard Exercise

Suppose I roll a fair 4-sided die, and simultaneously toss a fair coin.

(a) How many elements are in the outcome space?

(b) Let E denote the event "the coin lands heads." What is $\mathbb{P}(E)$?

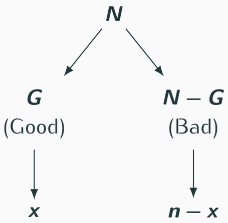
(c) What is the probability that the die lands on an even number?

(d) What is the probability that either the coin lands heads, or the die lands on an even number (or both?)

(By the way, we will often admit the "or both" in "or" statements like in part (d) above; that is, if we say "X or Y" we implicitly mean "X or Y or both")

Neat Diagrammatic Trick: Trees

Consider N objects: G of which are "good" and $N - G$ of which are "bad." In a sample of n, the probability of observing x "good" is



$$\Longrightarrow \frac{\binom{G}{x}\binom{N-G}{n-x}}{\binom{N}{n}}$$

Exercise: A box contains 20 marbles: 5 blue, 10 red, and 5 orange. I take a sample of size 5; what is the chance that I see 1 blue, 2 red, and 2 orange? (See Chalkboard)

More in Section!

- You will get some more practice with counting during Discussion Section this week, as well as in your first Homework.
- I could go on and on about counting! But, so that we don't get too burnt out during these first few lectures, I think I will put a pin on counting for now. We may need to introduce a few additional counting tricks/techniques throughout this course, but hopefully with the basics now mastered these additional techniques won't be anything too jarring!