7: Double Integral Review

PSTAT 120A: Summer 2022

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July 12, 2022

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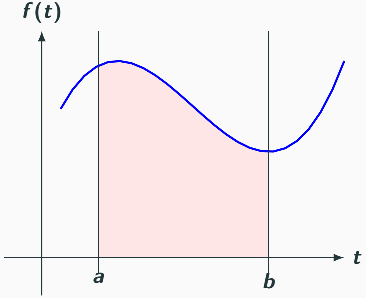
Leadup

Single Integrals; Simple Interval

- **Geometry of Single Integrals:**

$$\int_a^b f(x)\, \mathrm{d}x = \text{Area under } f(x)$$

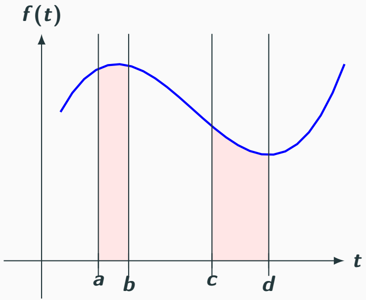


Single Integrals; More Complicated Interval

- **Geometry of Single Integrals:**

$$\int_R f(x)\,\mathrm{d}x = \text{Area under } f(x),$$

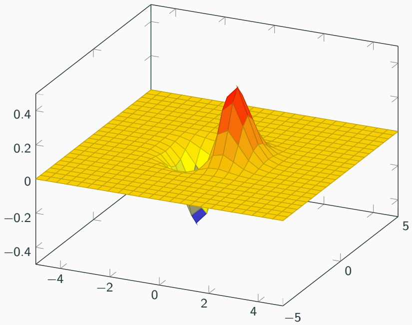
above the region $\boldsymbol{R}$



$$\boldsymbol{R = [a, b) \cup (c, d)}$$

Bivariate Functions

- The graph of a univariate function f is a one-dimensional curve, lying in $\mathbb{R}^2$.
- What does the graph of a bivariate function, $f(x, y)$ look like?
  - A **surface**, lying in $\mathbb{R}^3$.



Geometry of Double Integrals

- Since **single** integrals represent the **area** underneath the graph of a **univariate** function, it makes sense that a **double** integral represents the **volume** underneath the graph of a **bivariate** function.
- But, as we saw even in the univariate case; the region over which we integrate is important! In fact, with double integrals, often times the region of integration is the most important part of our computations!
- Let's talk a bit about regions.

Regions in $\mathbb{R}^2$

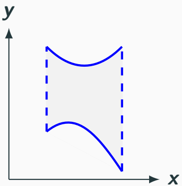
Four Types of Regions

By a "region" in $\mathbb{R}^2$, we mean a subset $\mathcal{R}$ of $\mathbb{R}^2$. This is also sometimes called a "domain," in the context of double integration.

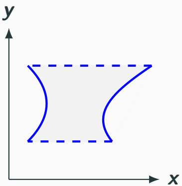
There are 4 different types of regions:

- Type I: y is bounded by two continuous function f_1 and f_2 of x, and x is bound on either side by a constant.
- Type II: x is bounded by two continuous function g_1 and g_2 of y, and y is bound on either side by a constant.
- Type III: both Type I and Type II
- Type IV: none of the above types.

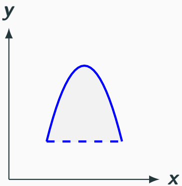
Four Types of Regions



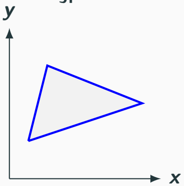
Type I



Type II



Type III



Type IV: Neither I, II, nor III

More General Regions

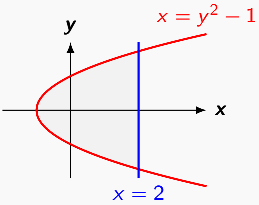
Of course, regions will typically be specified in some sort of mathematical format. Since, as I mentioned above, regions are simply subsets, we can use **set-builder notation.**

For example, a type I region may be expressed as

$$\mathcal{R} = \{(x, y) : g_1(x) \leq y \leq g_2(x) \ , \ a \leq x \leq b\}$$

Example 1

Example



- As a Type I region,

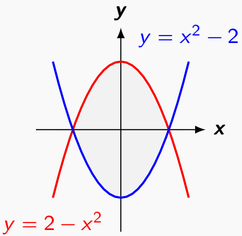
$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : -\sqrt{1+x} \leq y \leq \sqrt{1+x},$$
$$-1 \leq x \leq 2\}$$

- As a Type II region,

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : y^2 - 1 \leq x \leq 2,$$
$$-\sqrt{3} \leq y \leq \sqrt{3}\}$$

Example 2

Example



- As a Type I region,

$$\mathcal{R} = \{(x,y) : x^2 - 2 \le y \le 2 - x^2 \,, -\sqrt{2} \le x \le \sqrt{2}\}$$

- $\mathcal{R}$ cannot be expressed as a single Type II region, but rather as a union of two type II regions:

$$\mathcal{R} = \{(x,y) : -\sqrt{2+y} \le x \le \sqrt{2+y} \,,\ 0 \le y \le 2\}$$
$$\cup \{(x,y) : -\sqrt{2-y} \le x \le \sqrt{2-y} \,,\ -2 \le y \le 0\}$$

- Therefore, $\mathcal{R}$ is Type I.

Back to Double Integrals

Interpretation of the Double Integral

We therefore arrive at the following interpretation:

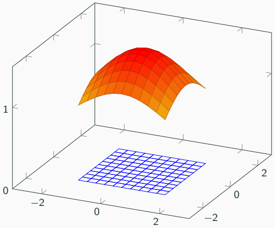
Point

Double integrals represent the **volume** underneath the portion of the graph of a bivariate function. That is,

$$\iint_{\mathcal{R}} f(x, y) \, \mathrm{d}A$$

represents the volume underneath the portion of the graph of $f(x, y)$ on the domain $\{(x, y) : (x, y) \in \mathcal{R}\}$.

Interpretation of the Double Integral



For example, if $\mathcal{R} \subset \mathbb{R}^2$ is a rectangle lying in the $xy-$ plane, then $\iint_{\mathcal{R}} f(x, y)\, \mathrm{d}y$ represents the volume of the prism with $\mathcal{R}$ as its bottom and $\{f(x, y) : (x, y) \in \mathcal{R}\}$ as its top (see figure to the left).

Double Integration over Rectangular Regions

First Result: Fubini's Theorem

Theorem: Fubini's Theorem

Consider the rectangle $\mathfrak{R} = \{(x, y) : a \leq x \leq b \ , \ c \leq y \leq d\}$ for $a, b, c, d \in \mathbb{R}$. If a function $f(x, y)$ is continuous on $\mathfrak{R}$, then

$$\iint_{\mathfrak{R}} f(x, y) \, \mathrm{d}A = \int_a^b \int_c^d f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_c^d \int_a^b f(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{1}$$

Sometimes, you will see the shorthand notation

$$\mathfrak{R} = [a, b] \times [c, d]$$

to refer to the definition of $\mathfrak{R}$ listed above. (Yup- this is our good friend, the Cartesian Product!)

Order of Integration

Note that one of the expressions in Fubini's theorem has a $\mathrm{d}x\ \mathrm{d}y$, and the other has a $\mathrm{d}y\ \mathrm{d}x$. The order in which these appear in the integral is sometimes called the **order of integration**.

This terminology stems from the *manner* in which we evaluate integrals: when we write, for example,

$$\int_c^d \int_a^b f(x, y)\ \mathrm{d}x\ \mathrm{d}y$$

we mean two things:

1. First evaluate $\int_a^b f(x, y)\ \mathrm{d}x$, treating y as a constant.
2. Integrate your result from step (1) above over y, letting y range from c to d.

Example

If $\mathfrak{R} = [0,1] \times [0,1]$ and $f(x,y) = x + y$, we have

$$\begin{aligned}
\iint_{\mathfrak{R}} f(x,y)\,\mathrm{d}A &= \int_0^1 \int_0^1 (x+y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \left(\int_0^1 (x+y)\,\mathrm{d}x \right)\,\mathrm{d}y \\
&= \int_0^1 \left[\frac{1}{2}x^2 + xy \right]_{x=0}^{x=1}\,\mathrm{d}y \\
&= \int_0^1 \left(\frac{1}{2} + y \right)\,\mathrm{d}y \\
&= \left[\frac{1}{2}y + \frac{1}{2}y^2 \right]_{y=0}^{y=1} = \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

Theorem: Theorem 2

Suppose $\mathfrak{R} = [a, b] \times [c, d]$ where a, b, c, and d are *constants* (i.e. do not depend on x or y). Then

$$\iint_{\mathfrak{R}} [f(x)g(y)] \, \mathrm{d}A = \left(\int_a^b f(x) \, \mathrm{d}x \right) \cdot \left(\int_c^d g(y) \, \mathrm{d}y \right) \tag{2}$$

In other words, if the integrand can be broken into the product of a function depending *only* on x and a function depending *only* on y, and if the limits of integration are constant, then the double integral can be evaluated as a product of single integrals.

Example 4

Example

If $\mathfrak{R} = [1, 2] \times [3, \pi]$ and $f(x, y) = x^2 y$, we have

$$\begin{aligned}
\iint_{\mathfrak{R}} f(x, y)\ \mathrm{d}A &= \int_3^{\pi} \int_1^2 x^2 y\ \mathrm{d}x\ \mathrm{d}y \\
&= \left(\int_1^2 x^2\ \mathrm{d}x \right) \cdot \left(\int_3^{\pi} y\ \mathrm{d}y \right) \\
&= \left[\frac{1}{3} x^3 \right]_{x=1}^{x=2} \cdot \left[\frac{1}{2} y^2 \right]_{y=3}^{y=\pi} \\
&= \frac{1}{3}(8 - 1) \cdot \frac{1}{2}(\pi^2 - 9) = \boxed{\frac{7(\pi^2 - 9)}{6}}
\end{aligned}$$

Double Integrals over General Regions

Another Theorem

Theorem: Theorem 3

If $\mathcal{D} = \{(x,y) : f(y) \le x \le g(y)\ ,\ c \le y \le d\}$, then

$$\iint_{\mathcal{D}} f(x,y)\ \mathrm{d}A = \int_c^d \int_{f(y)}^{g(y)} f(x,y)\ \mathrm{d}x\ \mathrm{d}y \tag{3}$$

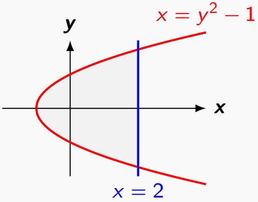
If $\mathcal{D} = \{(x,y) : h(x) \le y \le \ell(x)\ ,\ a \le x \le b\}$, then

$$\iint_{\mathcal{D}} f(x,y)\ \mathrm{d}A = \int_a^b \int_{h(x)}^{\ell(x)} f(x,y)\ \mathrm{d}y\ \mathrm{d}x \tag{4}$$

Example 5

Example

Let $\mathcal{R}$ be



and set $f(x, y) = xy$. If we express $\mathcal{R}$ as a Type II region (as we did at the beginning of Example 1), we may use 4 to write

$$\iint_{\mathcal{R}} f(x, y)\, \mathrm{d}A = \int_{-1}^{2} \int_{-\sqrt{1+x}}^{\sqrt{1+x}} xy \, \mathrm{d}y \, \mathrm{d}x$$

$$= \int_{-1}^{2} \left[\frac{1}{2} x y^2 \right]_{y=-\sqrt{1+x}}^{y=\sqrt{1+x}} \mathrm{d}x$$

$$= \frac{1}{2} \int_{-1}^{2} [x(1+x) - x(1+x)] \, \mathrm{d}x = \frac{1}{2} \int_{-1}^{2} (0) \, \mathrm{d}x = \boxed{0}$$

Example 5 (cont'd)

Let's see what happens if we express $\mathcal{R}$ as a Type I region instead:

$$\begin{aligned}
\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}A &= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{y^2-1}^{2} xy\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{2}\left[x^2 y\right]_{x=y^2-1}^{x=2}\,\mathrm{d}y \\
&= \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} [2y - y(y^2-1)^2]\,\mathrm{d}y \\
&= \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} (2y - y^5 - 2y^3 + y)\,\mathrm{d}y \\
&= \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} (3y - y^5 - 2y^3)\,\mathrm{d}y \\
&= \frac{1}{2}\left[\frac{3}{2}y^2 - \frac{1}{6}y^6 - y^3\right]_{y=-\sqrt{3}}^{y=\sqrt{3}} \\
&= \frac{1}{2}\left[\frac{9}{2} - \frac{3}{2} - 3\sqrt{3} - \frac{9}{2} + \frac{3}{2} + 3\sqrt{3}\right] = \boxed{0}
\end{aligned}$$

Example 5 (cont'd)

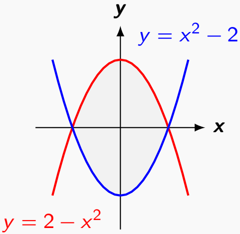
Clearly, one order of integration was significantly simpler than the other!

We can take this one step further:

Example 6

Example

Let $\mathcal{R}$ be



and let $f(x,y)$ be an arbitrary integrable function. Since $\mathcal{R}$ can be expressed as a single Type I region, we may write

$$\iint_{\mathcal{R}} f(x,y)\ \mathrm{d}A = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2-2}^{2-x^2} f(x,y)\ \mathrm{d}y\ \mathrm{d}x$$

Example

However, also recall that $\mathcal{R}$ cannot be expressed as a *single* Type II region, but rather the union of two Type II regions. As such, define

$$\mathcal{R}_1 := \{(x,y) : -\sqrt{2-y} \le x \le \sqrt{2-y}\,,\ -2 \le y \le 0\}; \quad \mathcal{R}_2 := \{(x,y) : -\sqrt{2+y} \le x \le \sqrt{2+y}\,,\ 0 \le y \le 2\};$$

so that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ and

$$\iint_{\mathcal{R}} f(x,y)\ \mathrm{d}A = \iint_{\mathcal{R}_1 \cup \mathcal{R}_2} f(x,y)\ \mathrm{d}A$$

$$= \int_{-\sqrt{2}}^{0} \int_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y)\ \mathrm{d}x\ \mathrm{d}y + \int_{0}^{\sqrt{2}} \int_{-\sqrt{2+y}}^{\sqrt{2+y}} f(x,y)\ \mathrm{d}x\ \mathrm{d}y$$

That is, if we integrate w.r.t. x first **we need to break our integral into the sum of two integrals!** Thus, it is clearly far less work to integrate with respect to y first.

Summary

Summary

Point: Summary:

- There are several types of regions in $\mathbb{R}^2$; Type I, Type II, Type III, and Type IV (everything else).
- Double integrals represent the volume underneath the graph of a bivariate function.
- Double integrals may be computed as two nested single integrals.
- There are always two different "orders of integration" when evaluating a Double Integral; in some cases, one order of integration is significantly simpler than the other.