EXAMPLES FROM SLIDE DECK 9

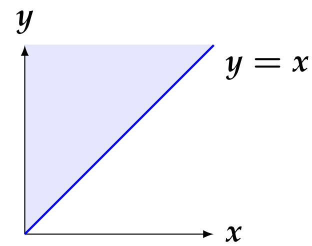
PSTAT 120A: Summer 2022 Instructor: Ethan P. Marzban

1. Suppose (X, Y) is a pair of random variables with joint density given by

$$f_{X,Y}(x,y) = \begin{cases} c \cdot e^{-(x+y)} & \text{if } x \leq y < \infty,\ 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the value of c that ensures $f_{X,Y}(x,y)$ is a valid joint p.d.f..

Solution: As always, we begin by sketching the support:



We now select the value of c that ensures the joint p.d.f. integrates to unity, when integrated over the support. For this particular setup, neither order of integration is significantly more difficult than the other. If we used dy dx, then we compute

$$\begin{aligned}\int_0^\infty \int_x^\infty ce^{-(x+y)} \, \mathrm{d}y \, \mathrm{d}x &= \int_0^\infty ce^{-x} \left(\int_x^\infty e^{-y} \, \mathrm{d}y \right) \, \mathrm{d}x \\ &= c \int_0^\infty e^{-x} \cdot e^{-x} \, \mathrm{d}x = c \int_0^\infty e^{-2x} \, \mathrm{d}x = \frac{c}{2} \stackrel{!}{=} 1 \implies \boxed{c = 2}\end{aligned}$$

If instead we had used dx dy, we would have

$$\begin{aligned}\int_0^\infty \int_0^y ce^{-(x+y)} \, \mathrm{d}x \, \mathrm{d}y &= \int_0^\infty ce^{-y} \left(\int_0^y e^{-x} \, \mathrm{d}x \right) \, \mathrm{d}y \\ &= c \int_0^\infty e^{-y} \cdot \left(1 - e^{-y}\right) \, \mathrm{d}x = c \int_0^\infty \left(e^{-y} - e^{-2y}\right) \mathrm{d}y \\ &= c \left(1 - \frac{1}{2}\right) = \frac{c}{2} \stackrel{!}{=} 1 \implies \boxed{c = 2}\end{aligned}$$

As an Aside: I personally like to set up these types of integrals, whenever possible, to include an ∞ in the upper limit of integration, especially when the integrand contains exponential functions. This is because $e^{-\infty} = 0$ which often simplifies things a bit.

   (b) Compute $\mathbb{P}(X \geq 0.5,\ Y \geq 0.5)$

Solution: When we write $\mathbb{P}(X \geq 0.5,\ Y \geq 0.5)$ we really mean

$$\mathbb{P}\left((X, Y) \in \{(X, Y) : X \geq 0.5,\ Y \geq 0.5\}\right)$$

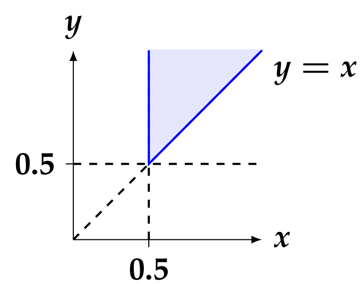
meaning

$$\mathbb{P}(X \geq 0.5,\ Y \geq 0.5) = \iint_{\{(x,y): x \geq 0.5,\ y \geq 0.5\}} f_{X,Y}(x, y)\ \mathrm{d}x$$

Now, we know that $f_{X,Y}(x, y) = 0$ whenever (x, y) is not in the support of (X, Y). In other words, the integrand above is not nonzero over the entire region $\{(x, y) : x \geq 0.5,\ y \geq 0.5\}$ but rather only over the *intersection* of $\{(x, y) : x \geq 0.5,\ y \geq 0.5\}$ and the support. Therefore, the integral above is equivalent to computing

$$\iint_{\mathcal{R}} 2e^{-(x+y)}\ \mathrm{d}A$$

where $\mathcal{R}$ is the region



Once again, either order of integration is fine. Using $\mathrm{d}y\ \mathrm{d}x$ we have

$$\begin{aligned}
\mathbb{P}(X + Y \geq 2) &= \int_{0.5}^{\infty} \int_{x}^{\infty} 2e^{-(x+y)}\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_{0.5}^{\infty} 2e^{-x} \int_{x}^{\infty} e^{-y}\ \mathrm{d}y\ \mathrm{d}x = \int_{0.5}^{\infty} 2e^{-2x}\ \mathrm{d}x = e^{-0.5 \cdot 2} = \boxed{e^{-1}}
\end{aligned}$$

Using $\mathrm{d}x\ \mathrm{d}y$, we have

$$\begin{aligned}
\mathbb{P}(X + Y \geq 2) &= \int_{0.5}^{\infty} \int_{0.5}^{y} 2e^{-(x+y)}\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_{0.5}^{\infty} 2e^{-y} \int_{0.5}^{y} e^{-x}\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_{0.5}^{\infty} 2e^{-y}\left(e^{-0.5} - e^{-y}\right)\ \mathrm{d}y \\
&= 2\int_{0.5}^{\infty} \left(e^{-(y+0.5)} - e^{-2y}\right)\ \mathrm{d}y \\
&= 2e^{-0.5} \int_{0.5}^{\infty} e^{-y}\ \mathrm{d}y - \int_{0.5}^{\infty} 2e^{-2y}\ \mathrm{d}y \\
&= 2e^{-0.5} \cdot e^{-0.5} - e^{-2 \cdot 0.5} = 2e^{-1} - e^{-1} = \boxed{e^{-1}}
\end{aligned}$$

(c) Compute $\mathbb{E}[XY]$

Solution: We return to the region sketched in part (a) [i.e. the support]. The multivariate analog of the LOTUS tells us

$$\mathbb{E}[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y) f_{X,Y}(x,y) \, \mathrm{d}A$$

meaning, using $g(x,y) = xy$, we have

$$\mathbb{E}[XY] = \iint_{\mathbb{R}^2} xy f_{X,Y}(x,y) \, \mathrm{d}A$$

Using $\mathrm{d}y \, \mathrm{d}x$ we have

$$\begin{aligned}
\mathbb{E}[XY] &= \int_0^\infty \int_x^\infty xy \cdot 2e^{-(x+y)} \, \mathrm{d}y \, \mathrm{d}x \\
&= 2 \int_0^\infty xe^{-x} \int_x^\infty ye^{-y} \, \mathrm{d}y \, \mathrm{d}x \\
&= 2 \int_0^\infty xe^{-x} \left[-e^{-y}(y+1) \right]_{y=x}^{y=\infty} \, \mathrm{d}x \\
&= 2 \int_0^\infty xe^{-x} e^{-x}(x+1) \, \mathrm{d}x \\
&= 2 \int_0^\infty (x^2 e^{-2x} + xe^{-2x}) \, \mathrm{d}x \\
&= 2 \left[\frac{\Gamma(3)}{2^3} \int_0^\infty \frac{2^3}{\Gamma(3)} \cdot x^{3-1} e^{-2x} \, \mathrm{d}x + \frac{\Gamma(2)}{2^2} \int_0^\infty \frac{2^2}{\Gamma(2)} \cdot x^{2-1} e^{-2x} \, \mathrm{d}x \right] \\
&= 2 \left[\frac{\Gamma(3)}{2^3} + \frac{\Gamma(2)}{2^2} \right] \\
&= \frac{\Gamma(3)}{4} + \frac{\Gamma(2)}{2} = \frac{2}{4} + \frac{1}{2} = \boxed{1}
\end{aligned}$$

The order $\mathrm{d}x \, \mathrm{d}y$ is a bit more tedious, so I recommend avoiding using that order for this particular part.

(d) Find $f_X(x)$, the marginal density of X.

Solution: To find the density of X, we integrate out y. The question becomes: what should the limits of our integral be? Well, it is true that

$$f_{X,Y}(x,y) = \int_{-\infty}^\infty f_{X,Y}(x,y) \, \mathrm{d}y$$

it's just that the integrand is zero for a significant portion of the the interval over which we are integrating!

Let me demonstrate the detailed way of thinking about this- it involves indicators. We can write

$$f_{X,Y}(x,y) = 2e^{-(x+y)} \cdot \mathbb{1}_{\{(x,y): y \geq x,\ x \geq 0\}}$$

Note that we can rewrite the density as

$$f_{X,Y}(x,y) = 2e^{-(x+y)} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot \mathbb{1}_{\{y \geq x\}}$$

Therefore,

$$\begin{aligned} f_{X,Y}(x,y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ \mathrm{d}y \\ &= \int_{-\infty}^{\infty} 2e^{-x}e^{-y} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot \mathbb{1}_{\{y \geq x\}}\ \mathrm{d}y \end{aligned}$$

We can safely pull everything involving only x outside of the integral:

$$\begin{aligned} f_{X,Y}(x,y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ \mathrm{d}y \\ &= \int_{-\infty}^{\infty} 2e^{-x}e^{-y} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot \mathbb{1}_{\{y \geq x\}}\ \mathrm{d}y \\ &= 2e^{-x} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot \int_{-\infty}^{\infty} e^{-y} \mathbb{1}_{\{y \geq x\}}\ \mathrm{d}x \end{aligned}$$

Now it is perhaps clearer what our limits of integration are; the integrand is nonzero only when $y \geq x$, meaning

$$\begin{aligned} f_{X,Y}(x,y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ \mathrm{d}y \\ &= \int_{-\infty}^{\infty} 2e^{-x}e^{-y} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot \mathbb{1}_{\{y \geq x\}}\ \mathrm{d}y \\ &= 2e^{-x} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot \int_{-\infty}^{\infty} e^{-y} \mathbb{1}_{\{y \geq x\}}\ \mathrm{d}x \\ &= 2e^{-x} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot \int_{x}^{\infty} e^{-y}\ \mathrm{d}y = 2e^{-2x} \cdot \mathbb{1}_{\{x \geq 0\}} \end{aligned}$$

which actually allows us to recognize $X \sim \mathrm{Exp}(2)$.

One benefit of this method of thinking about the problem is that we see that the support of $\underline{X}$ is also built into our computations!

(e) Find $f_Y(y)$, the marginal density of Y.

Solution: Let's play the same game as we did in part (d) above. Now, however, we will rewrite the joint as

$$f_{X,Y}(x,y) = 2e^{-(x+y)} \cdot \mathbb{1}_{\{(x,y):0 \leq x \leq y,\ y \geq 0\}}$$

(try and convince yourself that this is in fact the same density as before! All I've done is view $\mathcal{R}$ as a Type II region as opposed to a Type I region.) Now we compute

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} 2e^{-x}e^{-y} \cdot \mathbb{1}_{\{0 \leq x \leq y\}} \cdot \mathbb{1}_{\{y \geq 0\}}\, \mathrm{d}x \\ &= 2e^{-y} \cdot \mathbb{1}_{\{y \geq 0\}} \cdot \int_{-\infty}^{\infty} e^{-x} \cdot \mathbb{1}_{\{0 \leq x \leq y\}}\, \mathrm{d}x \\ &= 2e^{-y} \cdot \mathbb{1}_{\{y \geq 0\}} \int_0^y e^{-x}\, \mathrm{d}x = 2e^{-y}\left(1 - e^{-y}\right) \cdot \mathbb{1}_{\{y \geq 0\}} \end{aligned}$$

(f) Are X and Y independent? Explain.

Solution: There are a few acceptable explanations. The first one utilizes the definition of independence: we see that

$$\begin{aligned} f_{X,Y}(x,y) &= 2e^{-2x} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot 2e^{-y}\left(1 - e^{-y}\right) \cdot \mathbb{1}_{\{y \geq 0\}} \\ &= 4e^{-(x+y)} \cdot \mathbb{1}_{\{(x,y):x \geq 0,\ y \geq 0\}} \\ &\neq 2e^{-(x+y)} \cdot \mathbb{1}_{\{(x,y):y \geq x,\ x \geq 0\}} = f_{X,Y}(x,y) \end{aligned}$$

which shows X and Y are NOT independent. The other justification stems from noting that the support is nonrectangular, and therefore X and Y must be dependent.

(g) Compute $\text{Cov}(X, Y)$.

Solution: Recall that

$$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y] = (1) - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = 1 - \frac{3}{4} = \frac{1}{4}$$

(I leave it to you to figure out how I computed $\mathbb{E}[X]$ and $\mathbb{E}[Y]$).

(h) Compute $\text{Var}(X - Y)$

Solution: In general,

$$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y)$$

Plugging in $a = 1$ and $b = -1$ yields

$$\begin{aligned}\text{Var}(X - Y) &= (1)^2\text{Var}(X) + (-1)^2\text{Var}(Y) + 2(1)(-1)\text{Cov}(X, Y) \\ &= \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y) = \frac{1}{4} + \frac{5}{4} - 2\left(\frac{1}{4}\right) = \boxed{1}\end{aligned}$$

(I leave it to you to figure out how I computed $\text{Var}(X)$ and $\text{Var}(Y)$).

2. Consider a pair (X, Y) of discrete random variables with joint p.m.f. given by

$X \backslash Y$	1	2	3	4
0	0.1	0.1	0.1	0
1	0	0.2	0.1	0.1
2	0.1	0.1	0	0.1

(a) Find the marginal p.m.f.'s $p_X(x)$ and $p_Y(y)$ of X and Y respectively.

Solution: We compute the row- and column-sums

$X \backslash Y$	1	2	3	4	
0	0.1	0.1	0.1	0	0.3
1	0	0.2	0.1	0.1	0.4
2	0.1	0.1	0	0.1	0.3
	0.2	0.4	0.2	0.2	1

This allows us to read off the marginal p.m.f.'s of X and Y:

k	0	1	2
$p_X(k)$	0.3	0.4	0.3

;

k	1	2	3	4
$p_Y(k)$	0.2	0.4	0.2	0.2

(b) Compute $\mathbb{E}[XY]$.

Solution: By the LOTUS,

$$
\begin{aligned}
\mathbb{E}[XY] &= \sum_{x=0}^{2}\sum_{y=1}^{4} xy p_{X,Y}(x,y) \\
&= (0)(1)p_{X,Y}(0,1) + (0)(2)p_{X,Y}(0,2) + (0)(3)p_{X,Y}(0,3) + (0)(4)p_{X,Y}(0,4) \\
&\quad + (1)(1)p_{X,Y}(1,1) + (1)(2)p_{X,Y}(1,2) + (1)(3)p_{X,Y}(1,3) + (1,4)p_{X,Y}(1,4) \\
&\quad + (2)(1)p_{X,Y}(2,1) + (2)(2)p_{X,Y}(2,2) + (2)(3)p_{X,Y}(2,3) + (2)(4)p_{X,Y}(2,4) \\
&= (0)(1)(0.1) + (0)(2)(0.1) + (0)(3)(0.1) + (0)(4)(0) \\
&\quad + (1)(1)(0) + (1)(2)(0.2) + (1)(3)(0.1) + (1)(4)(0.1) \\
&\quad + (2)(1)(0.1) + (2)(2)(0.1) + (2)(3)(0) + (2)(4)(0.1) = 2.5
\end{aligned}
$$

(c) Are X and Y independent? Explain.

Solution: If X and Y were independent, then we would have

$$p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y) \quad \forall (x,y) \in \{0,1,2\} \times \{1,2,3,4\}$$

However, we see

$$p_X(1) \cdot p_Y(1) = (0.4)(0.2) = 0.08 \neq 0 = p_{X,Y}(1,1)$$

Hence, this is enough for us to conclude that X and Y are NOT independent. By the way, $(1,1)$ was not the only point we could have used; there are several other points (x,y) for which $p_{X,Y}(x,y) \neq p_X(x) \cdot p_Y(y)$. But, to establish dependence, we need only one such point.

3. Let $X_1, \cdots, X_n$ be a sequence of random variables with the following covariance structure:

$$\text{Cov}(X_i, X_j) = \begin{cases} 1 & \text{if } i = j \\ 0.5 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute $\text{Var}\left(\sum_{i=1}^{n} X_i\right)$

Solution: In general,

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} X_i + 2\sum_{i<j} \text{Cov}(X_i, X_j)$$

Let's focus on this second sum. We know that, by construction, $\mathrm{Cov}(X_i, X_j) = 0$ whenever i and j are more than 1 units away from each other. Additionally, whenever i and j are exactly 1 unit away from each other the covariance is simply 0.5. Therefore, we have some number of copies of (0.5); the exact number of copies is the number of indices i and j such that both i and j are in the set $\{1, \cdots, n\}$ **and** i and j are one unit apart from each other. Upon inspection, we see there are $(n-1)$ such indices, meaning

$$\sum_{i<j} \mathrm{Cov}(X_i, X_j) = (n-1)(0.5)$$

and so

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i \right) = n(1) + 2(n-1)(0.5) = n + n - 1 = \boxed{2n-1}$$

If the argument for computing $\sum_{i<j} \mathrm{Cov}(X_i, X_j)$ is a bit too abstract, we can explicitly construct a table which displays $\mathrm{Cov}(X_i, X_j)$ for all indices i and j:

$$\begin{array}{cc|cccccc}
 & & & & (\boldsymbol{i}) & & & \\
 & & 1 & 2 & 3 & 4 & \cdots & n \\
\hline
 & 1 & 1 & 0.5 & 0 & 0 & \cdots & n \\
 & 2 & 0.5 & 1 & 0.5 & 0 & \cdots & 0 \\
(\boldsymbol{j}) & 3 & 0 & 0.5 & 1 & 0.5 & \cdots & 0 \\
 & 4 & 0 & 0 & 0.5 & 1 & \cdots & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 & n & 0 & 0 & 0 & 0 & \cdots & 1
\end{array}$$

Here we can explicitly see that there are precisely $(n-1)$ terms equal to 0.5 that lie below the diagonal (remember, elements below the diagonal correspond to indicies (i, j) where $i < j$).