Score: ___________ / **90**

PSTAT 120A / **FINAL EXAM** / Summer 2022 Instructor: **Ethan Marzban**

NAME: ______________________________________ **Perm Number:** ______________________

SECTION (circle one): **3:30 - 4:20pm (Lucas)** **5 - 5:50pm (Moya)** **8 - 8:50am (Moya)**

Instructions:

- You will have **170 minutes** to complete this exam.
- You are allowed the use of two **8.5 $\times$ 11-inch** sheets, front and back, of notes. You are also permitted the use of **calculators**; the use of any and all other electronic devices (laptops, cell phones, etc.) is prohibited.
- Unless otherwise specified, simplification is not needed; however, all integrals and infinite sums (unless otherwise specified) must be evaluated.
  - One exception is that, whenever applicable, answers may be left in terms of Φ, the standard normal c.d.f..
- Good Luck!!!

Honor Code: In signing my name below, I certify that all work appearing on this exam is entirely my own and not copied from any external source. I further certify that I have not received any unauthorized aid while taking this exam.

$\times$ ____

Multiple Choice Questions:

Question:	1	2	3	Total
Points:	1	1	1	3
Score:				

Short-Answer Questions:

Question:	4	5	6	7	8	9	10	11	12	13	Total
Points:	7	9	11	11	10	9	5	11	9	5	87
Score:											

1 Multiple Choice Questions

Please fill in the bubble(s) ***on the exam below*** *corresponding to your answer. You do not need to submit any additional work for these questions.*

1. For $(X,Y) \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and a fixed constant $z \in [0,1]$, which of the following correctly provides $\mathbb{P}(X+Y \leq z)$? [1pts.]

   ◯ $\int_0^z \int_0^{z-x} (1)\ \mathrm{d}x\ \mathrm{d}y$

   ◯ $\int_0^1 \int_0^{z-x} (1)\ \mathrm{d}x\ \mathrm{d}y$

   ◯ $\int_0^z \int_0^{z-y} (1)\ \mathrm{d}x\ \mathrm{d}y$

   ◯ $\int_0^1 \int_0^{z-y} (1)\ \mathrm{d}x\ \mathrm{d}y$

   ◯ None of the above

2. Given a bivariate random vector (X,Y) with joint probability density function (p.d.f.) given by $f_{X,Y}(x,y)$, which of the following correctly computes $\mathbb{E}[\cos(X+Y)]$? (Only one answer choice is correct.) [1pts.]

   ◯ $\int_{-\infty}^{\infty} \cos(x+y) f_{X,Y}(x,y)\ \mathrm{d}x$

   ◯ $\iint_{\mathbb{R}^2} \cos(x+y) f_X(x) f_Y(y)\ \mathrm{d}A$

   ◯ $\iint_{\mathbb{R}^2} \cos(x+y) f_{X,Y}(x, z-x)\ \mathrm{d}A$

   ◯ $\iint_{\mathbb{R}^2} \cos(x+y) f_{X,Y}(x,y)\ \mathrm{d}A$

   ◯ None of the above

3. A hand of 5 cards is dealt without replacement from a standard deck of 52 cards. What is the probability that the hand contains exactly 2 pairs (by "pair" we mean "matching in <u>denomination</u>, not <u>suit</u>.")? [1pts.]

- ◯ $\dfrac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$

- ◯ $\dfrac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{(52)_5}$

- ◯ $\dfrac{\binom{13}{4}\binom{4}{1}^4\binom{12}{1}\cdot(5!)}{\binom{52}{5}}$

- ◯ $\dfrac{\binom{13}{4}\binom{4}{1}^4\binom{12}{1}\cdot(5!)}{(52)_5}$

- ◯ None of the above

2 Short Answer Questions

Please mark your final answers in the spaces provided below each question. ***Be sure to show all of your work!***

4. The archipelago of *Gauchonia* is a collection of 12 smaller islands and one "Main Island," on which the capital lies. A ferry exists to help locals commute from island to island. On one particular trip, 10 people board the ferry at the capital on the Main Island, and then request to disembark on a random island, independently of all other passengers. Assume the following:

   - Nobody disembarks at the Main Island
   - Nobody can stay on the ferry forever (i.e. everyone disembarks at *some* island)
   - The ferry only stops at an island if someone is disembarking at that island.

   Let X denote the number of islands at which the ferry makes a stop.

   (a) What is $\mathbb{P}(X = 12)$? [2pts.]

   (b) Compute $\mathbb{E}[X]$. As a hint: Elevator Problem. [5pts.]

5. Let (X, Y) be a bivariate random vector with joint probability density function (p.d.f.) given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{6}{5} \cdot xy & \text{if } \sqrt{y} \leq x \leq 2,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $f_{X,Y}(x, y)$ is a valid joint p.d.f. **Be sure to show ALL of your steps fully;** large lapses in logic will result in point penalties. [4pts.]

(b) **Set up, but DO NOT EVALUATE** a double integral (or set of double integrals) corresponding to $\mathbb{P}(X + Y \leq 2)$. [3pts.]

(c) Are X and Y independent? Justify your answer. [2pts.]

6. A point is selected in the first quadrant such that the $x-$ and $y-$ coordinates of the point are independent and both follow the distribution with probability density function (p.d.f.) given by

$$f(t) = \begin{cases} 2\lambda t e^{-\lambda t^2} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$ is a constant.

(a) Show that $f(t)$ is a valid p.d.f. [3pts.]

(b) Identify the distribution (by name!!) of X^2 (remember that X, the $x-$coordinate of the point, follows the distribution with p.d.f. given above). **Include any/all relevant parameter(s).** [5pts.]

(c) Identify the distribution (by name!!) of $X^2 + Y^2$. **Include any/all relevant parameter(s).** As a reminder, you can use previously-derived results without proof so long as you cite them. [3pts.]

7. Let (X, Y) be a bivariate random vector with joint probability density function (p.d.f.) given by

$$f_{X,Y}(x, y) = \begin{cases} \lambda y e^{-y(x+\lambda)} & \text{if } x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$ is a fixed constant.

(a) Find $f_Y(y)$, the marginal p.d.f. of Y and use this to identify Y as belonging to a known distribution. **Be sure to include any/all relevant parameter(s)!** [4pts.]

(b) Find $f_{X|Y}(x \mid y)$, the density of $(X \mid Y = y)$, and use this to identify $(X \mid Y = y)$ as belonging to a known distribution. **Be sure to include any/all relevant parameter(s)!** [3pts.]

(c) Set up **but do not evaluate** an integral corresponding to $\mathbb{E}[X]$, that only involves the marginal p.d.f. of Y. [4pts.]

8. In a class of highly motivated PSTAT 120A students, the average time taken for a randomly selected student to complete the final exam is 30 minutes.

   (a) Use Markov's Inequality to bound the probability that a randomly selected student will complete the exam in under 31 minutes. **Be sure to clearly state whether this is an upper or lower bound!** [3pts.]

(b) Suppose that it is also known that the standard deviation of the time a randomly selected students takes to complete the exam is 5 minutes. Use Chebyshev's Inequality to bound the probability that a randomly selected student will complete the exam in under 31 minutes. **Be sure to clearly state whether this is an upper or lower bound!** [3pts.]

(c) Now, suppose 64 students are sampled at random (and with replacement), and the average time to complete the final exam among these 64 students is recorded. Approximately what is the probability that the average completion time was lower than 31 minutes? [4pts.]

9. Doctor Strange is travelling through the Multiverse, visiting parallel universes one by one. There is, however, a 25% chance that any one of his trips will cause an Incursion, independently of all of his other trips.

   (a) What is the probability that exactly 3 of the first 5 trips Doctor Strange makes result in Incursions? [3pts.]

   (b) What is the probability that the 7^{th} trip Doctor Strange makes is the first trip that results in an Incursion? [3pts.]

(c) What is the expected number of trips Doctor Strange must make before causing his third Incursion? [3pts.]

10. At a boutique store, 60% of items are *Gaucci* brand. Additionally, 60% of all items sold at this boutique are found to be counterfeits; 50% of these counterfeit items are *Gaucci* brand. Suppose you purchase an item that is *Gaucci* brand-what is the probability that your item is counterfeit? [5pts.]

11. Suppose (X, Y) is a discrete bivariate random vector with joint probability mass function (p.m.f.) given by

		Y = 1	2	3
X	0	0.1	0.2	0.1
	1	0	0.2	0
	2	0.4	0	0

Furthermore, define events A, B, C in the following manner (recall that 0 is an even number):

$$A = \{X \text{ is even}\}$$
$$B = \{Y \text{ is odd}\}$$
$$C = \{Y \text{ is a multiple of } 3\}$$

(a) Compute $\mathbb{P}(A)$ [3pts.]

(b) Compute $\mathbb{P}(A \cup B)$ [4pts.]

(c) Are A, B, C mutually independent? Justify your answer <u>mathematically</u>, using the definition of mutual independence. [4pts.]

12. The speed of a randomly selected car travelling along I-5 is a random variable X with MGF (Moment-Generating Function) given by

$$M_X(t) = \exp\left\{70t + \frac{40}{2}t^2\right\}; \quad \forall t \in \mathbb{R}$$

Additionally, the posted speed limit is 65mph.

(a) What is the average speed of a randomly selected car travelling along I-5? [2pts.]

(b) What is the probability that a randomly selected car will be speeding? (Here, “speeding” means “travelling faster than the speed limit”) [3pts.]

(c) Given that a car was speeding, what is the chance that it was travelling slower than 72 mph? [4pts.]

13. Consider a random variable X with mean μ and variance σ^2. If f is an appropriately differentiable function, use a second-order Taylor Series Expansion (i.e. a Taylor Series Expansion taken out to the third term in the sum) to show that [5pts.]

$$\mathbb{E}[f(X)] \approx f(\mu) + \frac{\sigma^2}{2} f''(\mu)$$

COMMON MGF's:

Distribution	MGF at t
$\text{Bin}(n,p)$	$(1-p+pe^t)^n, \quad \forall t \in \mathbb{R}$
$\text{Geom}(p)$	$\begin{cases} \dfrac{pe^t}{1-(1-p)e^t} & \text{if } t < -\ln(1-p) \\ \infty & \text{otherwise} \end{cases}$
$\text{NegBin}(r,p)$	$\begin{cases} \left(\dfrac{pe^t}{1-(1-p)e^t}\right)^r & \text{if } t < -\ln(1-p) \\ \infty & \text{otherwise} \end{cases}$
$\text{Pois}(\lambda)$	$e^{\lambda(e^t-1)}, \quad \forall t \in \mathbb{R}$

Distribution	MGF at t
$\text{Exp}(\lambda)$	$\begin{cases} \dfrac{\lambda}{\lambda-t} & \text{if } t < \lambda \\ 0 & \text{otherwise} \end{cases}$
$\text{Gamma}(r,\lambda)$	$\begin{cases} \left(\dfrac{\lambda}{\lambda-t}\right)^r & \text{if } t < \lambda \\ 0 & \text{otherwise} \end{cases}$
$\mathcal{N}(\mu,\sigma^2)$	$\exp\left\{\mu t + \dfrac{\sigma^2}{2}t^2\right\}; \quad \forall t \in \mathbb{R}$
$\text{Unif}[a,b]$	$\begin{cases} \dfrac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$

USEFUL RESULTS FROM MATHEMATICS:

(***Please note that these are by no means comprehensive; I expect you to know all of the sums from the Calculus Review Series, as well as common mathematical results.***)

- $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
- $\cosh(x) := \dfrac{e^x + e^{-x}}{2}; \quad \sinh(x) := \dfrac{e^x - e^{-x}}{2}$
- $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + (x-a)f'(a) + \dfrac{1}{2}(x-a)^2 f''(a) + \dfrac{1}{3!}(x-a)^3 f'''(a) + \cdots$

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