Total Points: 10

PSTAT 120A / **QUIZ 3 SOLUTIONS** / Summer 2022 Instructor: **Ethan Marzban**

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1. (1 point) **Multiple Choice** Cars arrive at a tollbooth according to a Poisson Process at a rate of 3 cars every minute. Which of the following statements is true?
   - ◯ The probability that we must wait exactly 3 minutes between the 4^{th} and 6^{th} arrivals is $\frac{3^2}{\Gamma(2)} \cdot 3^{2-1} \cdot e^{-3 \cdot 3}$
   - ◯ On average, we expect 4 cars to arrive every 2 minutes
   - ◯ On average, we expect to wait 4 minutes between the arrival of the 3^{rd} car and the 5^{th} car.
   - √ **None of the above are correct.**

2. Let $X \sim \mathcal{N}(3, 5)$. Compute each of the following, **leaving your answers in terms of Φ wherever necessary**

   (a) (2 points) $\mathbb{P}(X < 4)$

   Solution: We standardize, and then plug into Φ:

   $$\mathbb{P}(X < 4) = 1 - \mathbb{P}\left(\frac{X-3}{\sqrt{5}} < \frac{4-3}{\sqrt{5}}\right) = \Phi\left(\frac{1}{\sqrt{5}}\right)$$

   (b) (2 points) $\mathbb{E}[X^2]$

   Solution:

   $$\mathbb{E}[X^2] = \text{Var}(X) + [\mathbb{E}(X)]^2 = 5 + (3^2) = 14$$

   - +1pt for recognizing $\mathbb{E}[X^2] = \text{Var}(X) + [\mathbb{E}(X)]^2$
   - +1pt for correct final answer [consistent with part (a); that is, if students used $\text{SD}(X) = 5$ in part (a) and also in part (b), do not double-deduct on part (b)]

3. (5 points) Let X be a random variable with the following probability density function (p.d.f.):

   $$f_X(x) = \begin{cases} |x| & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

   Also, define the random variable Y as $Y := X^2$. Find $f_Y(y)$, the p.d.f. of Y.

Solution: Note that $S_Y = [0,1]$.

Method 1: The C.D.F. Method For $y \in [0,1]$, we have

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(|X| \le \sqrt{y}) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y})$$
$$= \int_{-\sqrt{y}}^{\sqrt{y}} |x| \, \mathrm{d}x = 2\int_0^{\sqrt{y}} x \, \mathrm{d}x = 2 \cdot \frac{1}{2}(\sqrt{y})^2 = y$$

We could at this point differentiate $F_Y(y)$ w.r.t. y to find $f_Y(y)$ directly, or we could recognize this as the c.d.f. of the Unif$[0,1]$ distribution; in either case, we find

$$f_Y(y) = \begin{cases} 1 & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Method 2: The Change of Variable Formula We could use the change of variable formula as well, but we would need to split the state space into two subregions:

- $\underline{S_X^{(1)} = [-1,0]}$: For $x \in [-1,0]$ we have $g(x) = x^2$ and $g^{-1}(y) = -\sqrt{y}$, meaning

  $$\left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| = \left| -\frac{1}{2\sqrt{y}} \right| = \frac{1}{2\sqrt{y}}$$

  and so we have

  $$f_X^{(1)}(y) = \frac{1}{2\sqrt{y}} \cdot |-\sqrt{y}| \cdot \mathbb{1}_{\{y \in [0,1]\}} = \frac{y}{2} \cdot \mathbb{1}_{\{y \in [0,1]\}}$$

  (where we have noted that the region $S_X^{(1)}$ gets mapped to $[0,1]$ under g.)

- $\underline{S_X^{(2)} = [-1,0]}$: For $x \in [0,1]$ we have $g(x) = x^2$ and $g^{-1}(y) = \sqrt{y}$, meaning

  $$\left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| = \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{2\sqrt{y}}$$

  and so we have

  $$f_X^{(2)}(y) = \frac{1}{2\sqrt{y}} \cdot |\sqrt{y}| \cdot \mathbb{1}_{\{y \in [0,1]\}} = \frac{y}{2} \cdot \mathbb{1}_{\{y \in [0,1]\}}$$

  (where we have noted that the region $S_X^{(2)}$ also gets mapped to $[0,1]$ under g.)

Therefore, putting everything together: for $y \in [0,1]$ we have

$$f_Y(y) = f_Y^{(1)}(y) + f_Y^{(2)}(y) = \frac{y}{2} \cdot \mathbb{1}_{\{y \in [0,1]\}} + \frac{y}{2} \cdot \mathbb{1}_{\{y \in [0,1]\}} = y \cdot \mathbb{1}_{\{y \in [0,1]\}}$$

or, equivalently,

$$f_Y(y) = \begin{cases} 1 & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$