Total Points: 10

PSTAT 120A / **QUIZ 4 SOLUTIONS** / Summer 2022 Instructor: **Ethan Marzban**

1. (1 point) **Multiple Choice:** I hope you've been paying attention this quarter... Where have our lectures been held?

   √ **PSYCH 1902**
   ◯ BROIDA 1610
   ◯ SOUTH HALL 1430
   ◯ SOUTH HALL 5421
   ◯ None of the other answer choices

2. Let $X, Y \overset{\text{i.i.d.}}{\sim} \text{Unif}[0,1]$

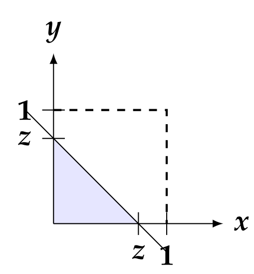
   (a) (1 point) What is $f_{X,Y}(x,y)$, the joint probability density function (p.d.f.) of (X,Y)? **Justify your answer**.

   Solution: By independence, $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$. We also know that $f_X(x) = \mathbb{1}_{\{0 \leq x \leq 1\}}$ and $f_Y(y) = \mathbb{1}_{\{0 \leq y \leq 1\}}$; therefore,

   $$f_{X,Y}(x,y) = \mathbb{1}_{\{0 \leq x \leq 1,\ 0 \leq y \leq 1\}} = \begin{cases} 1 & \text{if } x \in [0,1],\ y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

   (b) (2 points) If $z \in [0,1]$, compute $\mathbb{P}(X + Y \leq z)$. **USE INTEGRALS; don't just use geometry.** Additionally, **Evaluate all integrals fully, and sketch the region of integration for full credit.**

   Solution: If $z \in [0,1]$, our picture looks like:



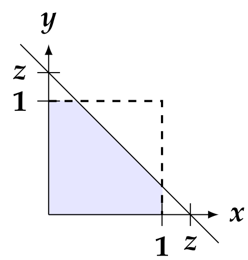
   Either order of integration is fine; using $\mathrm{d}x\ \mathrm{d}y$, we find

   $$\begin{aligned} \mathbb{P}(X+Y \leq z) &= \int_0^z \int_0^{z-y} (1)\ \mathrm{d}x\ \mathrm{d}y \\ &= \int_0^z (z-y)\ \mathrm{d}y = \left[zy - \frac{1}{2}y^2 \right]_{y=0}^{y=z} = z^2 - \frac{1}{2}z^2 = \frac{1}{2}z^2 \end{aligned}$$

(It would have been acceptable to also compute the area of the shaded triangle using geometry, without the need for any integrals.)

(c) (2 points) If $z \in [1, 2]$, compute $\mathbb{P}(X + Y \leq z)$. For this part, **set up BUT DO NOT EVALUATE** the integral. For full points, you must still sketch the region of integration.

Solution: If $z \in [1, 2]$, our picture looks like:



The points of intersection between the line and the boundaries $x = 1$, $y = 1$ are $(1, z-1)$ and $(z-1, 1)$. Again, either order of integration is fine; using dx dy, we find

$$\mathbb{P}(X + Y \leq z) = \int_0^{z-1} \int_0^1 (1) \, \mathrm{d}x \, \mathrm{d}y + \int_{z-1}^1 \int_0^{z-y} (1) \, \mathrm{d}x \, \mathrm{d}y$$

3. Suppose X is a random variable with MGF (Moment-Generating Function) given by

$$M_X(t) = \begin{cases} \dfrac{e^t}{1-2t} & \text{if } t < \dfrac{1}{2} \\ \infty & \text{otherwise} \end{cases}$$

(a) (2 points) Compute $\mathbb{E}[X]$. **Show all of your work for full credit!**

Solution: There are two ways to solve this problem. The way I anticipate most people will use to solve this is using the fact that

$$\mathbb{E}[X] = M_X'(0)$$

Thus, we first differentiate $M_X(t)$, and then evaluate at $t = 0$:

$$M_X'(t) = \frac{e^t(1-2t) + 2e^t}{(1-2t)^2}$$
$$\mathbb{E}[X] = M_X'(0) = \frac{e^0(1 - 2 \cdot 0) + 2e^0}{(1 - 2 \cdot 0)^2} = \frac{1+2}{1} = 3$$

....

The other, slightly more esoteric way to solve this problem, is to let $Y \sim \text{Exp}(1)$ so that

$$M_Y(t) = \begin{cases} \frac{1}{1-t} & \text{if } t < 1 \\ \infty & \text{otherwise} \end{cases}$$

Also, recall that $M_{aY+b}(t) = e^{bt} M_Y(at)$. Thus, note that if $X = 2Y + 1$, we have

$$M_{2Y+1}(t) = e^t \cdot \begin{cases} \frac{1}{1-2t} & \text{if } 2t < 1 \\ \infty & \text{otherwise} \end{cases} = \begin{cases} \frac{e^t}{1-2t} & \text{if } t < \frac{1}{2} \\ \infty & \text{otherwise} \end{cases}$$

which is indeed the MGF of X. Hence, $X = 2Y + 1$ and, since $\mathbb{E}[Y] = 1$,

$$\mathbb{E}[X] = \mathbb{E}[2Y + 1] = 2\mathbb{E}[Y] + 1 = 2(1) = 1 = \boxed{3}$$

(b) (2 points) Suppose we have another random variable $Y \sim \mathcal{N}(3, 2)$ that is independent of X. If we define $Z := X + Y$, find $M_Z(t)$, the MGF of Z. **Be sure to specify the values of t over which $M_Z(t)$ is infinite!**

Solution: Since $X \perp Y$, we know that $M_Z(t) = M_X(t)M_Y(t)$. We were provided $M_X(t)$; we can also lookup/recall that $M_Y(t) = \exp\{2t + t^2\}$. Additionally, $M_X(t) < \infty$ only when $t < 1/2$; thus, fix a $t < 1/2$ so that

$$M_Z(t) = M_X(t)M_Y(t) = \frac{e^t}{1-2t} \cdot e^{2t} \cdot e^{t^2} = \frac{e^{3t+t^2}}{1-2t}$$

and hence the full MGF of Z is

$$M_Z(t) = \begin{cases} \dfrac{e^{3t+t^2}}{1-2t} & \text{if } t < 1/2 \\ \infty & \text{otherwise} \end{cases}$$