PSTAT 120A, Summer 2022: Practice Problems 1

Week 1

WELCOME TO PSTAT 120A! Discussion worksheets are designed to give you additional practice with material covered in lecture the week prior. Please note that this worksheet contains material from Wednesday's lecture of Week 1.

A note on the "Extra Problems:" though these may not get covered in section, it is in your best interest to complete them as they are still fair game for quizzes/exams!

Conceptual Review

(a) What type of mathematical object (i.e. function, set, constant, etc.) is the **outcome space**?

(b) What type of mathematical object (i.e. function, set, constant, etc.) is an **event**?

(c) What are the **Axioms of Probability**?

Problem 1: Component Failures *(modified from Ross 2.5)*

A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

a) How many outcomes are in the sample space of this experiment?

b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

d) Write out all the outcomes in the event AW.

e) Compute $\mathbb{P}(A \cup W)$, assuming equally likely outcomes.

Solution:

a) The outcome space is expressible as:

$$\Omega = \left\{(x_1, x_2, x_3, x_4, x_5) : x_i \in \{0, 1\}, \ i = 1, 2, 3, 4, 5\right\} = \{0, 1\}^5$$

Therefore, by a result from lecture,

$$|\Omega| = |\{0, 1\}^5| = \left(|\{0, 1\}|\right)^5 = 2^5 = 32$$

b) By enumeration, we find

$$\begin{aligned} W = \big\{ & (1,1,1,1,1)\ ,\ (1,1,1,1,0)\ ,\ (1,1,1,0,1)\ , (1,1,0,1,1)\ ,\ (1,1,1,0,0)\ , \\ & (1,1,0,1,0)\ ,\ (1,1,0,0,1)\ ,\ (1,1,0,0,0)\ ,\ (1,0,1,1,1)\ ,\ (0,1,1,1,1)\ , \\ & (1,0,1,1,0)\ ,\ (0,1,1,1,0)\ ,\ (0,0,1,1,1)\ ,\ (0,0,1,1,0)\ ,\ (1,0,1,0,1) \big\} \end{aligned}$$

c) We could, in theory, list out the outcomes in A and count them. However, we can also utilize our counting techniques to avoid this. We have no restrictions on the first three components; there are therefore $2^3 = 8$ possible states of "working" or "not working" for these three. But, we require the final two components to be non-working, meaning there is only one choice for their statuses. In other words, the total number of outcomes in A is

$$2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 8$$

d) In words, the event AW means "the system is working, **and** components 4 and 5 are not working." The only way for this to happen is if components 1 and 2 are both working (as both of the other scenarios in which the system works involves either component 4 or component 5 to be functional, which is not the case.) Therefore,

$$AW = \{(1,1,0,0,0),\ (1,1,1,0,0)\}$$

e) By the Inclusion-Exclusion Rule:

$$\mathbb{P}(A \cup W) = \frac{|A| + |W| - |AW|}{2^5} = \frac{8 + 15 - 2}{32} = \frac{21}{32} = 0.65625$$

By the Way: As was mentioned in lecture, sometimes we drop the intersection symbol and write AB in place of $A \cap B$. So, the event AW referenced in parts (d) and (e) is the same thing as the event $A \cap W$.

Problem 2: Poker Face

Suppose a hand of 5 cards is drawn from a standard deck of 52 playing cards. Compute the probabilities of the following hands:

a) Full House (3 of a kind and a 2 of a kind)

b) Three-of-a-kind

c) Two pairs (i.e. two distinct two-of-a-kinds)

Solution: For each part, we count the number of favorable outcomes and divide by $\binom{52}{5}$, the total number of 5-card hands.

a)
- $\binom{13}{1}$: choose the rank [from the 13 possible ranks] of the three-of-a-kind
- $\binom{4}{3}$: choose the suits of the three-of-a-kind
- $\binom{12}{1}$: choose the rank [from the remaining 12] of the two-of-a-kind
- $\binom{4}{2}$: choose the suits of the two-of-a-kind

$$\implies \boxed{\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}}$$

b)
- $\binom{13}{1}$: choose the rank [from the 13 possible ranks] of the three-of-a-kind
- $\binom{4}{3}$: choose the suits of the three-of-a-kind
- $\binom{12}{2}$: choose the two **distinct** ranks [from the remaining 12] to constitute the non-special cards
- $\binom{4}{1} \cdot \binom{4}{1}$: choose the two non-special cards

$$\implies \boxed{\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}}$$

c)
- $\binom{13}{2}$: choose the ranks [from the 13 possible ranks] of the two pairs
- $\binom{4}{2}^2$: choose the two pairs
- $\binom{44}{1}$: from the viable $52 - 4 - 4 = 44$ cards, select one to be the non-special card. Where did the two -4's come from? Well, let's consider a specific hand: $(A_S, A_D, K_C, K_H, \_\_\_)$ where A_S means "ace of spades," etc. and the blank represents the final card. If the blank space were another ace, we would have a full house an *not* a two-pair hand. The same is true if the final card were a king. Therefore, the rank of the final card must be different than the ranks of the two paired cards; in other words, there are only $52 - 4 - 4$ viable cards.

$$\implies \boxed{\frac{\binom{13}{2}\binom{4}{2}^2\binom{44}{1}}{\binom{52}{5}}}$$

Extra Problems

Problem 3: Further Practice with Sums

From the Calculus Review video, we saw the result:

$$\sum_{k=0}^{\infty} kr^k = \frac{r}{(1-r)^2}; \qquad |r| < 1 \tag{1}$$

a) Using a similar method used to derive (1), derive an expression for $\sum_{k=a}^{\infty} kr^k$. You may once again assume that the infinite sum and the derivative operator interchange.

b) Compute $E := \sum_{\substack{k=0 \\ \text{even}}}^{\infty} kr^k$ and $O := \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} kr^k$. *Hint: Reindex the sums.*

c) Which is larger, O or E? (Or are they the same value?) In other words, when summing over only the even natural numbers do we obtain a result that is equal to, larger than, or less that the result we would have obtained had we summed over only the *odd* natural numbers?

Solution:

a) We begin with the result

$$\sum_{k=a}^{\infty} r^k = \frac{r^a}{1-r}$$

Differentiating both sides w.r.t. r (and, as the problem statement suggests, assuming the derivative passes through the infinite sum on the RHS), we find

$$\begin{aligned} \sum_{k=a}^{\infty} kr^{k-1} &= \frac{ar^{a-1}(1-r) + r^a}{(1-r)^2} \\ &= \frac{r^a\left[ar^{-1}(1-r) + 1\right]}{(1-r)^2} \\ \sum_{k=a}^{\infty} kr^k &= r \times \frac{r^a\left[ar^{-1}(1-r) + 1\right]}{(1-r)^2} \\ &= \boxed{\frac{r^a\left[a(1-r) + r\right]}{(1-r)^2}} \end{aligned}$$

again, of course, assuming $|r| < 1$.

b) In both cases, we reindex the sum. When computing the sum over the even natural numbers we set $k = 2n$, so that

$$\begin{aligned}
\sum_{\substack{k=0 \\ \text{even}}}^{\infty} k r^k &= \sum_{n=0}^{\infty} (2n) r^{2n} \\
&= 2 \sum_{n=0}^{\infty} n (r^2)^n \\
&= 2 \times \frac{r^2}{(1-r^2)^2} = \boxed{\frac{2r^2}{(1-r^2)^2}}
\end{aligned}$$

When computing the sum over the odd natural numbers we set $k = 2n + 1$ so that

$$\begin{aligned}
\sum_{\substack{k=1 \\ \text{odd}}}^{\infty} k r^k &= \sum_{n=0}^{\infty} (2n+1) r^{2n+1} \\
&= 2r \sum_{n=0}^{\infty} n (r^2)^n + r \sum_{n=0}^{\infty} (r^2)^n \\
&= r \left[2 \cdot \frac{r^2}{(1-r^2)^2} + \frac{1}{1-r^2} \right] \\
&= r \left[\frac{2r^2 + 1 - r^2}{(1-r^2)^2} \right] = \boxed{\frac{r(r^2+1)}{(1-r^2)^2}}
\end{aligned}$$

with the implicit assumption that both results hold for $|r| < 1$. As a quick check:

$$\begin{aligned}
\frac{2r^2}{(1-r^2)^2} + \frac{r(r^2+1)}{(1-r^2)^2} &= \frac{2r^2 + r^3 + r}{(1-r^2)^2} \\
&= \frac{r(r^2 + 2r + 1)}{(1-r^2)^2} \\
&= \frac{r(r+1)^2}{[(1-r)(1+r)]^2} = \frac{r}{(1-r)^2}
\end{aligned}$$

c) We see that

$$\frac{E}{O} = \frac{2r^2}{r(r^2+1)} = \frac{2r}{r^2+1}$$

From a graph, one can verify that $|2r/(r^2 + 1)| < 1$ [one can also differentiate, find the extreme values, and show that the second derivative behaves appropriately so as to ensure these extrema correspond to a min and a max at $r = \pm 1$, respectively]. In other words, $\boxed{O > E}$.

Problem 4 (Challenge): Castle Conundrum *(modified from Ross 2.17)*

A chessboard is a 8×8 board of 64 squares. Among the pieces on a chessboard is the **rook** (shaped like a castle); rooks can only move horizontally or vertically. If another chesspiece lies in the potential path of a rook, we say that the rook can "capture" the other chesspiece (see picture below).

Hint: Once you place a rook, you effectively "reduce" the size of the chessboard.

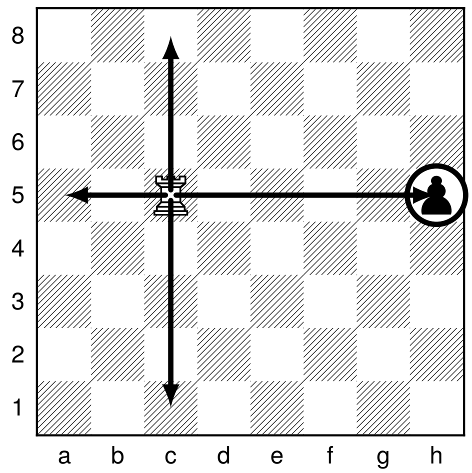

Fig. 1: *The pawn (circled) is in trouble...*

Suppose 8 rooks are to be placed on a chessboard. What is the chance that none of the 8 rooks can capture each other?

Solution: Once we place a rook, we lose one rank and one file in which to place the second rook. For instance, after placing our first rook there are only $7 \times 7 = 7^2$ possible squares in which to place the second rook. After placing the first two rooks, there are only $(6)^2$ possible squares in which to place the third rook. And so on and so forth. Therefore, the total number of ways to place the 8 rooks so that no rook is able to be captured by another is

$$\prod_{i=1}^{8} (i^2)$$

The total number of places to place the rooks, with no constraints, is $(64)_8$; therefore, assuming equally likely outcomes, the desired probability is

$$\frac{\prod_{i=1}^{8}(i^2)}{(64)_8} = \frac{(8!)^2}{64 \times 63 \times \cdots \times 57} = \frac{8!}{\binom{64}{8}} \approx 9.109 \times 10^{-6}$$