PSTAT 120A, Summer 2022: Practice Problems 2

Week 2

Conceptual Review

(a) Intuitively, what does $\mathbb{P}(A \mid B)$ represent?

(b) What is the definition of independence? What is the intuition behind this definition?

(c) Does pairwise independence imply mutual independence? Does mutual independence imply pairwise independence?

(d) What type of mathematical object is a Random Variable?

Problem 1: Proving Independence

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two events $A, B \in \mathcal{F}$. Show that if $A \perp B$, then $A^\complement \perp B^\complement$.

Solution: We write

$$\begin{aligned}
\mathbb{P}(A^\complement \cap B^\complement) &= \mathbb{P}[(A \cup B)^\complement] \\
&= 1 - [\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)] = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B) \\
&= 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A) \cdot \mathbb{P}(B) \\
&= 1 - \mathbb{P}(A) - \mathbb{P}(B)\, [1 - \mathbb{P}(A)] \\
&= [1 - \mathbb{P}(A)] \cdot [1 - \mathbb{P}(B)] \\
&= \mathbb{P}(A^\complement) \cdot \mathbb{P}(B^\complement)
\end{aligned}$$

Problem 2: Conditional Complements *(modified from ASV 2.7)*

a) Argue that $\{A^\complement \cap B\ ,\ A \cap B\}$ forms a partition of the event B.

b) Show that $\mathbb{P}(A^\complement \mid B) = 1 - \mathbb{P}(A \mid B)$.

c) Suppose $\mathbb{P}(A \mid B) = 0.6$ and $\mathbb{P}(B) = 0.5$. Find $\mathbb{P}(A^\complement \cap B)$.

d) Suppose now that $A \subseteq B$. Find a simple formula for $\mathbb{P}(A \mid B^\complement)$.

Hint: You can either use mathematical arguments, or sketch a Venn Diagram.

Solution:

a) Mathematically, we can see that

$$\begin{aligned}
(A^\complement \cap B) \cap (A \cap B) &= (A \cap A^\complement) \cap (B \cap B) = \varnothing \cap B = \varnothing \\
(A^\complement \cap B) \cup (A \cap B) &= [(A^\complement \cap B) \cup A] \cap [(A^\complement \cap B) \cup B] \\
&= [(A^\complement \cup A) \cap (A \cup B)] \cap [(A^\complement \cup B) \cap (B \cup B)] \\
&= (A \cup B) \cap (B) = B
\end{aligned}$$

which proves the desired result. [Note that in going from the second-to-last line to the last line we utilized the fact that $B \subseteq (A^\complement \cup B)$.] A Venn Diagram also yields the desired result. ∎

b) $\mathbb{P}(A^\complement \mid B) = \dfrac{\mathbb{P}(A^\complement \cap B)}{\mathbb{P}(B)}$

$$= \frac{\mathbb{P}(B) - \mathbb{P}(A \cap B)}{\mathbb{P}(B)} = 1 - \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = 1 - \mathbb{P}(A \mid B) \quad \blacksquare$$

Note that, by part (a), $\mathbb{P}(A^\complement \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

c) There are several ways to approach this problem. One is to write

$$\begin{aligned}\mathbb{P}(A^\complement \cap B) &= \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(B) - \mathbb{P}(A \mid B) \cdot \mathbb{P}(B) = 0.5 - 0.6 \cdot 0.5 = \boxed{0.2}\end{aligned}$$

Alternatively, we could note that by part (b)

$$\mathbb{P}(A^\complement \cap B) = \mathbb{P}(A^\complement \mid B) \cdot \mathbb{P}(B) = [1 - \mathbb{P}(A \mid B)] \cdot \mathbb{P}(B) = (1 - 0.6) \cdot 0.5 = \boxed{0.2}$$

d) Since $A \subseteq B$ we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)$ and

$$\mathbb{P}(A \mid B^\complement) = \frac{\mathbb{P}(A \cap B^\complement)}{\mathbb{P}(B^\complement)} = \frac{\mathbb{P}(A) - \mathbb{P}(A \cap B)}{\mathbb{P}(B^\complement)} = \frac{\mathbb{P}(A) - \mathbb{P}(A)}{\mathbb{P}(B^\complement)} = \boxed{0}$$

In hindsight, we could have guessed this answer: we are saying that A is completely contained inside B. If B has not occurred [as in, if $B^\complement$)], then it is impossible for A to have occurred.

Problem 3: Selecting Words

A word is selected at random from the sentence

`STATISTICS IS SO COOL`

Then, a letter is selected at random from the chosen word.

a) What is the probability that the letter "S" is selected?

b) If X denotes the length of the chosen word, what is the PMF of X?

c) Continuing from part (b); what is $\mathbb{E}[X]$?

d) Continuing from part (b); what is $\mathbb{E}\left[\frac{1}{X}\right]$?

e) For every vowel in your selected word, you are awarded \$1; for every consonant, however, you are forced to pay \$1. Letting W denote your net gain/loss, what is $\mathbb{E}[W]$?

Hint: Try finding the PMF of W first.

Solution:

a) Label the words 1 through 4; let W_i denote the event "word i was selected" and let S denote the event "the letter S was selected." We seek $\mathbb{P}(S)$; by the Law of Total Probability,

$$\begin{aligned}\mathbb{P}(S) &= \sum_{i=1}^{4} \mathbb{P}(S \mid W_i) \cdot \mathbb{P}(W_i) \\ &= \frac{3}{10} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{13}{40}\end{aligned}$$

Here is a bit more detail: firstly, since words are selected at random, $\mathbb{P}(W_i) = 1/4$ for every $i = 1, 2, 3, 4$. Now, what does $\mathbb{P}(S \mid W_1)$ represent? This is the probability that the letter S was selected after selecting the word `STATISTICS`. Since letters are also selected at random, this is simply the number of S's divided by the number of letters in `STATISTICS`; i.e. $3/10$.

b) Note that `STATISTICS` has length 10; `IS` and `SO` both have length 2; `COOL` has length 4. Therefore, under X, we have

$$\begin{aligned}\texttt{STATISTICS} &\mapsto 10 \\ \texttt{IS} &\mapsto 2 \\ \texttt{SO} &\mapsto 2 \\ \texttt{COOL} &\mapsto 4\end{aligned}$$

Since words are selected at random, each outcome is equally likely; hence

$$\mathbb{P}(X = 10) = \frac{1}{4}; \quad \mathbb{P}(X = 2) = \frac{2}{4}; \quad \mathbb{P}(X = 4) = \frac{1}{4}$$

c) From the definition of expected value,

$$\begin{aligned}\mathbb{E}[X] &= \sum_k k \cdot \mathbb{P}(X = k) \\ &= \sum_{k \in \{2,4,10\}} k \cdot \mathbb{P}(X = k) = (2)\left(\frac{2}{4}\right) + (4)\left(\frac{1}{4}\right) + (10)\left(\frac{1}{4}\right) = \frac{9}{2}\end{aligned}$$

d) In general,

$$\mathbb{E}[g(X)] = \sum_k g(k) \cdot \mathbb{P}(X = k)$$

Therefore, setting $g(k) = 1/k$ we have

$$\mathbb{E}\left[\frac{1}{X}\right] = \sum_{k \in \{2,4,10\}} \left(\frac{1}{k}\right) \cdot \mathbb{P}(X = k) = \left(\frac{1}{2}\right)\left(\frac{2}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{10}\right)\left(\frac{1}{4}\right) = \frac{27}{80}$$

e) Let's see what the mapping W looks like explicitly.

- The word `STATISTICS` has 7 consonants and 3 vowels; hence, `STATISTICS` $\mapsto 3 - 7 = -4$ under W.

- The word `IS` has 1 consonant and 1 vowel; hence, under W, `IS` $\mapsto 1 - 1 = 0$.
- The word `SO` has 1 consonant and 1 vowel; hence, under W, `SO` $\mapsto 1 - 1 = 0$.
- The word `COOL` has 2 consonants and 2 vowels; hence, under W, `COOL` $\mapsto 0$.

Again, since elements in the outcome space were equally likely we see

$$\mathbb{P}(W = 0) = \frac{3}{4}; \quad \mathbb{P}(W = -4) = \frac{1}{4}$$

and so, by the definition of expected value,

$$\mathbb{E}[W] = (0)\left(\frac{3}{4}\right) + (-4)\left(\frac{1}{4}\right) = -\$1$$

Extra Problems

Problem 4: Pólya's Urn Scheme

A box contains n marbles, b of which are blue and $g := n - b$ of which are gold. A marble is drawn at random and its color is noted; the marble is then placed back into the box along with k additional marbles of the same color (so now there are $n + k$ total marbles in the box). Now, another marble is drawn; find the probability that it is blue.

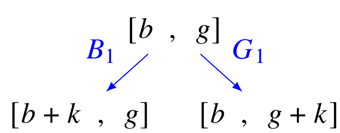
Solution: We begin by establishing the following notation:

$$\left.\begin{aligned} B_i &= \{i^{\text{th}}\text{marble drawn is blue}\} \\ G_i &= \{i^{\text{th}}\text{marble drawn is gold}\} \end{aligned}\right\} i = 1, 2$$

We seek the quantity $\mathbb{P}(B_2)$. Using the Law of Total Probability, we write this as

$$\mathbb{P}(B_2) = \mathbb{P}(B_2 \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(B_2 \mid G_1)\mathbb{P}(G_1)$$

To compute the conditional probabilities on the RHS, it may be helpful to visualize the configuration of the urn after each successive possibility:



That is,

$$\mathbb{P}(B_2 \mid B_1) = \frac{b + k}{n + k}; \quad \mathbb{P}(B_2 \mid G_1) = \frac{b}{n + k}$$

Therefore, continuing from the Law of Total Probability, we find

$$\begin{aligned} \mathbb{P}(B_2) &= \mathbb{P}(B_2 \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(B_2 \mid G_1)\mathbb{P}(G_1) \\ &= \frac{b + k}{n + k} \cdot \frac{b}{n} + \frac{b}{n + k} \cdot \frac{n - b}{n} \\ &= \frac{b(b + k) + b(n - b)}{n(n + k)} \\ &= \frac{b(\cancel{b} + k + n - \cancel{b})}{n(n + k)} = \frac{b\cancel{(n + k)}}{n\cancel{(n + k)}} = \boxed{\frac{b}{n}} \end{aligned}$$

Problem 5: Number of Conditions

Recall that when establishing the independence of n events, there are a series of computations we must perform (i.e. the "two-way intersections," "three-way intersections," etc.) Show that there are a total of $2^n - n - 1$ computations involved in establishing the independence of n events.

Hint: Recall the Binomial Theorem:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Solution: The number of "$k-$way intersections" is simply the number of ways to chose k of our total n events, without replacement and without regard to order. Thus, the number of "k-way intersections" is simply $\binom{n}{k}$. Thus, the total number of computations is simply

$$\sum_{k=2}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} - \binom{n}{0} - \binom{n}{1} = 2^n - 1 - n$$

(note that it makes no sense to talk about a "zero-way intersection," nor does it make sense to talk about a "one-way intersection".)

Problem 6: I Like to Prove It Prove It!

Prove the following:

(a) If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Hint: Partition $A \cup B$; then note that $A \cup B = B$ since $A \subseteq B$.

Solution: Note that $\{A, (B \setminus A)\}$ forms a partition of $A \cup B$. This means that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$$

Now, since $A \subseteq B$ we have that $A \cup B = B$; thus, we have shown that

$$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$$

Since $\mathbb{P}(B \setminus A) \geq 0$ by the first axiom of probability, we have that

$$\mathbb{P}(B) \geq \mathbb{P}(A)$$

thereby proving the desired result. ■

(b) If $A \subseteq B$, then $\mathbb{P}(B \mid A) = 1$. Provide both a mathematical proof, as well as an intuitive one.

Solution: For the mathematical proof, we write

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1$$

since, because $A \subseteq B$, we have $A \cap B = A$. For the intuitive argument: saying that A is a subset of B means that whenever A happens B is guaranteed to have happened. Since $\mathbb{P}(B \mid A)$ represents our updated beliefs on B in the presence of A, we must have that $\mathbb{P}(B \mid A) = 1$. ■

(c) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an event B with $\mathbb{P}(B) \neq 0$, the measure $\mathbb{P}_B(\cdot)$ defined through

$$\mathbb{P}_B(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

is a valid probability measure.

Hint: All you need to show is that $\mathbb{P}_B(\cdot)$ satisfies the three axioms of probability. Additionally, we know that $\mathbb{P}(\cdot)$ satisfies the axioms of probability.

Solution: Since $\mathbb{P}$ is a valid probability measure, it must satisfy the axioms of probability. In other words,

(1) $\mathbb{P}(A) \geq 0$ for every $A \in F$

(2) $\mathbb{P}(\Omega) = 1$

(3) For a sequence $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint events, $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

Now, we would like to show that $\mathbb{P}_B(\cdot)$ satisfies the axioms of probability as well.

(1) $\mathbb{P}_B(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$. Both $\mathbb{P}(A \cap B)$ and $\mathbb{P}(B)$ are nonnegative, by (1) above, meaning $\mathbb{P}_B(A) \geq 0$. ✓

(2) $\mathbb{P}_B(\Omega) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$, since $\Omega \cap B = B$. ✓

(3) For a sequence of pairwise disjoint events $\{A_i\}_{i=1}^{\infty}$, we have

$$\begin{aligned} \mathbb{P}_B\left(\bigcup_{i=1}^{\infty} A_i\right) &= \frac{\mathbb{P}_B(\bigcup_{i=1}^{\infty} A_i)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}[\bigcup_{i=1}^{\infty}(A_i \cap B)]}{\mathbb{P}(B)} \\ &= \frac{\sum_{i=1}^{\infty} \mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_{i=1}^{\infty} \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_{i=1}^{\infty} \mathbb{P}(A_i \mid B) \end{aligned}$$

Since $\mathbb{P}_B(\cdot)$ satisfies the three axioms of probability, is is a valid probability measure. ■