PSTAT 120A, Summer 2022: Practice Problems 3

Week 2

Conceptual Review

(a) What does expected value measure?

(b) What does variance measure?

Problem 1: Linearity of Expectation

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X with expectation $\mu := \mathbb{E}[X]$.

(a) Prove that $\mathbb{E}[aX + b] = a\mu + b$.

Solution: By the LOTUS with $g(k) = ak + b$, we find

$$\mathbb{E}[aX + b] = \sum_k (ak + b) p_X(k) = a \sum_k k p_X(k) + b \sum_k p_X(b)$$

We recognize $\sum_k k p_X(k) =: \mathbb{E}[X] =: \mu$, and also $\sum_k p_X(k) = 1$ meaning $\mathbb{E}[aX + b] = a\mu + b$, as desired. ■

(b) Prove that $\mathbb{E}[g(X) + h(X)] = \mathbb{E}[g(X)] + \mathbb{E}[h(X)]$

Solution: Another application of the LOTUS yields

$$\mathbb{E}[g(X) + h(X)] = \sum_k [g(k) + h(k)] p_X(k) = \sum_k g(k) p_X(k) + \sum_k h(k) p_X(k) = \mathbb{E}[g(X)] + \mathbb{E}[h(X)]$$

as desired. ■

Problem 2: Verifying P.M.F's

Let X be a random variable with p.m.f. given by

$$p_X(k) = \begin{cases} c & \text{if } k = 0 \\ \left(\frac{1}{3}\right)^k & \text{if } k = 1, 2, \cdots \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that ensures $p_X(k)$ is a valid probability measure.

Solution: We require $\sum_k p_X(k) = 1$, meaning we should probably first compute

$$\sum_{k=1}^{\infty} \left(\frac{1}{3} \right)^k = \frac{\left(\frac{1}{3}\right)}{1 - \left(\frac{1}{3}\right)} = \frac{1}{2}$$

Therefore, we require

$$c + \frac{1}{2} = 1 \implies c = \frac{1}{2}$$

(b) Compute the probability that X is even. (Recall that 0 is even.)

Solution: The key observation is that

$$\{X \text{ is even}\} = \bigcup_{\substack{k=0 \\ \text{even}}}^{\infty} \{X = k\}$$

Next, we take the probability of both sides; since the events on the RHS are all disjoint we can invoke the third axiom of probability to see

$$\begin{aligned} \mathbb{P}(X \text{ is even}) &= \sum_{\substack{k=0 \\ \text{even}}}^{\infty} p_X(k) \\ &= p_X(0) + \sum_{\substack{k=2 \\ \text{even}}}^{\infty} p_X(k) \\ &= \frac{1}{2} + \sum_{\substack{k=2 \\ \text{even}}}^{\infty} \left(\frac{1}{3} \right)^k \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^{2n} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{9} \right)^n = \frac{1}{2} + \frac{\left(\frac{1}{9}\right)}{1 - \left(\frac{1}{9}\right)} = \frac{5}{8} \end{aligned}$$

where in going from the second line to the third we have reindexed the sum with $k = 2n$.

(c) Compute $\mathbb{E}[X]$

Solution: From the definition of expected value,

$$\begin{aligned}\mathbb{E}[X] &= \sum_k k p_X(k) \\ &= (0)p_X(0) + \sum_{k=1}^{\infty} k \cdot \left(\frac{1}{3}\right)^k \\ &= \frac{\left(\frac{1}{3}\right)}{\left[1 - \left(\frac{1}{3}\right)\right]^2} = \boxed{\frac{3}{4}}\end{aligned}$$

where we have utilized the following identity, proven in the Calculus Review Video:

$$\sum_{k=0}^{\infty} k p^k = \frac{p}{(1-p)^2}, \quad |p| < 1$$

(d) It can be shown that $\mathbb{E}[X^2] = 3/2$. Compute $\text{Var}(X)$

Solution: From the definition of variance,

$$\text{Var}(X) = \mathbb{E}[X^2] - [\mathbb{E}(X)]^2 = \frac{6}{4} - \left(\frac{3}{4}\right)^2 = \boxed{\frac{15}{16}}$$

Problem 3: Proofreader

In a given book, each page contains a typo with probability 10% independently of all other pages. An editor begins examining the book page-by-page.

(a) What is the probability that among the first 10 pages examined the editor will find exactly 3 typos?

Solution: Let X denote the number of typos among the first 10 pages; then $X \sim \text{Bin}(10, 0.1)$ and

$$\mathbb{P}(X = 3) = \boxed{\binom{10}{3}(0.1)^3(0.9)^7}$$

(b) What is the probability that among the first 10 pages examined the editor will find at most 3 typos?

> **Solution:** Let X be defined as in part (a) above; we seek $\mathbb{P}(X \leq 3)$, which can be computed as
>
> $$\mathbb{P}(X \leq 3) = \sum_{k=0}^{3} \binom{10}{k} (0.1)^k (0.9)^{10-k} \approx \boxed{0.987}$$

(c) What is the probability that the 4th page the editor examines is the first page to contain a typo?

> **Solution:** Let Y denote the number of pages, including the final page, needed to observe the first typo; then $W \sim \text{Geom}(0.1)$ and
>
> $$\mathbb{P}(W = 4) = \boxed{(0.9)^3 \cdot (0.1)}$$

(d) What is the expected number of pages, including the final page, that the editor will need to examine before observing the first typo?

> **Solution:** Let Y be defined as in part (b) above; then
>
> $$\mathbb{E}[W] = \frac{1}{0.1} = \boxed{10}$$

Extra Problems

Problem 4: Variance of the Geometric Distribution

Let $X \sim \text{Geom}(p)$.

(a) Compute $\mathbb{E}[X(X-1)]$.

Hint: Try differentiating the geometric series repeatedly

Solution: First we compute

$$\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{k=0}^{\infty} p^k\right) = \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{1-p}\right)$$
$$\sum_{k=1}^{\infty} kp^{k-1} = \frac{1}{(1-p)^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{k=0}^{\infty} kp^{k-1}\right) = \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{(1-p)^2}\right)$$
$$\sum_{k=0}^{\infty} k(k-1)p^{k-2} = \frac{2}{(1-p)^3}$$
$$\sum_{k=0}^{\infty} k(k-1)p^{k} = \frac{2p^2}{(1-p)^3}$$

Now, returning to the random variable X, we compute [using the LOTUS]

$$\begin{aligned}\mathbb{E}[X(X-1)] &= \sum_k k(k-1)p_X(k) \\ &= \sum_{k=1}^{\infty} k(k-1)(1-p)^{k-1}\cdot p \\ &= \frac{p}{1-p}\sum_{k=0}^{\infty} k(k-1)(1-p)^k \\ &= \frac{p}{1-p} \times \frac{2(1-p)^2}{[1-(1-p)^3} \\ &= \boxed{\frac{2(1-p)}{p^2}}\end{aligned}$$

(b) Using your answer to part (a), find $\mathbb{E}[X^2]$.

Solution: Note that by the result of 1(a) above,

$$\mathbb{E}[X(X-1)] = \mathbb{E}[X^2 - X] = \mathbb{E}[X^2] - \mathbb{E}[X]$$

Therefore, rearranging terms,

$$\mathbb{E}[X^2] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] = \frac{2(1-p)}{p^2} + \frac{1}{p} = \frac{2-p}{p^2}$$

(c) Using your answer to part (b), show that $\text{Var}(X) = 1/p^2$.

Solution:

$$\text{Var}(X) = \mathbb{E}[X^2] - [\mathbb{E}(X)]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$