PSTAT 120A, Summer 2022: Practice Problems 5

Week 4

Conceptual Review

(a) Why is a function of a random variable also a random variable?

(b) If $Y := g(X)$ where the distribution of X is known, must we first find $f_Y(y)$ before computing $\mathbb{E}[Y]$?

(c) How do transformations of discrete random variables work?

(d) Will transformations of discrete random variables always be discrete? Will transformations of continuous random variables always be continuous?

Problem 1: Two Interesting Results

(a) If $X \sim \text{Exp}(\lambda)$ and $Y := cX$ for some fixed constant $c > 0$, show that $Y \sim \text{Exp}(\lambda/c)$. For practice, derive the result in two ways: using the c.d.f. method, and using the Change of Variable formula.

Solution: Using the CDF Method:

$$F_Y(y) := \mathbb{P}(Y \leq y) = \mathbb{P}(cX \leq y) = \mathbb{P}\left(X \leq \frac{y}{c}\right) = F_X\left(\frac{y}{c}\right)$$
$$= \begin{cases} 1 - e^{-\lambda\left(\frac{y}{c}\right)} & \text{if } \left(\frac{y}{c}\right) \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 - e^{-\left(\frac{\lambda}{c}\right)y} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which we recognize as the c.d.f. of a $\text{Exp}(\lambda/c)$ distribution.

Using the Change of Variable Formula: We take $g(t) = ct$ meaning $g^{-1}(t) = (t/c)$, and so

$$\left|\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y)\right| = \left|\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y}{c}\right)\right| = \left|\frac{1}{c}\right| = \frac{1}{c}$$

where we were able to drop the absolute value signs since $c > 0$ by assumption. Hence, by the Change of Variable formula, the nonzero portion of $f_Y(y)$ is given by

$$f_Y(y) = f_X\left[g^{-1}(y)\right] \cdot \left|\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y)\right| = \lambda e^{-\lambda\left(\frac{y}{c}\right)} \cdot \frac{1}{c} = \left(\frac{\lambda}{c}\right) e^{-\left(\frac{\lambda}{c}\right)y}$$

Coupled with the fact that $S_Y = [0, \infty)$ we have

$$f_Y(y) = \begin{cases} \left(\frac{\lambda}{c}\right) e^{-\left(\frac{\lambda}{c}\right)y} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which shows $Y \sim \text{Exp}(\lambda/c)$.

(b) If $X \sim \text{Gamma}(r, \lambda)$ and $Y := cX$ for some fixed constant $c > 0$, identify the distribution of Y **by name**, taking care to include any/all relevant parameter(s).

> **Solution: Using the CDF Method:**
>
> $$F_Y(y) := \mathbb{P}(Y \leq y) = \mathbb{P}(cX \leq y) = \mathbb{P}\left(X \leq \frac{y}{c}\right) = F_X\left(\frac{y}{c}\right)$$
>
> It will actually be inadvisable to try and simplify this integral any further, as the Gamma distribution does not (in general) have a simple closed-form expression for its C.D.F.. Instead, we can obtain the p.d.f. of Y directly by differentiating our expression above and utilizing the Chain Rule:
>
> $$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y} F_X\left(\frac{y}{c}\right) = \frac{1}{c} \cdot f_Y\left(\frac{y}{c}\right) \\ &= \frac{1}{c} \cdot \frac{\lambda^r}{\Gamma(r)} \left(\frac{y}{c}\right)^{r-1} \cdot e^{-\lambda\left(\frac{y}{c}\right)} \\ &= \frac{\left(\frac{\lambda}{c}\right)^r}{\Gamma(r)} y^{r-1} e^{-\left(\frac{\lambda}{c}\right) y} \end{aligned}$$
>
> which shows $Y \sim \text{Gamma}(r, \lambda/c)$. The Change of Variable formula would have functioned in much the same way for this problem.

Problem 2: Raise The Roof- er, Ceiling!

Let $X \sim \text{Exp}(\lambda)$, and define $Y := \lceil X \rceil$. Identify the distribution of Y **by name**, taking care to include any/all relevant parameter(s). Recall that

$$\lceil x \rceil := \text{smallest integer larger than or equal to } x$$

so, for instance, $\lceil \pi \rceil = 4$.

Hint: *Identify appropriate values for a and b such that*

$$\{\lceil X \rceil = y\} = \{a < X \leq b\}$$

> **Solution:** First note that the support of Y is $\{1, 2, 3, \dots\}$, meaning Y is discrete. Now, following the hint, we relate the p.m.f. of Y to the c.d.f. of X by writing
>
> $$p_Y(y) := \mathbb{P}(Y = y) = \mathbb{P}(\lceil X \rceil = y)$$
>
> Upon inspection, we note that
>
> $$\{\lceil X \rceil = y\} = \{y - 1 < X \leq y\}$$
>
> Thus, we have
>
> $$\begin{aligned} p_Y(y) &= \mathbb{P}(\lceil X \rceil = y) \\ &= \mathbb{P}(y - 1 < X \leq y) \\ &= F_X(y) - F_X(y - 1) \\ &= \cancel{1} - e^{-\lambda y} - \cancel{1} + e^{-\lambda(y-1)} \\ &= e^{-\lambda(y-1)} - e^{-\lambda y} \end{aligned}$$

$$
\begin{aligned}
&= e^{-\lambda y} \cdot e^{\lambda} - e^{-\lambda y} \\
&= e^{-\lambda y}(e^{\lambda} - 1) \\
&= e^{-\lambda y} e^{-\lambda}(1 - e^{-\lambda}) \\
&= \left(e^{-\lambda}\right)^{(y-1)} (1 - e^{-\lambda}) \\
&= \left[1 - (1 - e^{-\lambda})\right]^{y-1} \cdot \left(1 - e^{-\lambda}\right)
\end{aligned}
$$

showing that

$$Y \sim \mathrm{Geom}(1 - e^{-\lambda}) \text{ on } \{1, 2, 3, \dots\}$$

As an aside: The factorization for this problem may not come very naturally to most. That is, it may be tempting to write

$$\mathbb{P}(Y = y) = e^{-\lambda y}(e^{\lambda} - 1)$$

If you have an intuition that this might follow the Geometric distribution, but don't quite know what parameter it should follow, you can "cheat" by finding the expectation of Y directly:

$$
\begin{aligned}
\mathbb{E}(Y) &= \sum_{y=1}^{\infty} y \cdot e^{-\lambda y}(e^{\lambda} - 1) \\
&= (e^{\lambda} - 1) \cdot \sum_{y=1}^{\infty} y \left(e^{-\lambda}\right)^{y} \\
&= (e^{-\lambda} - 1) \cdot \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} \\
&= \cancel{(1 - e^{-\lambda})} \cdot \frac{1}{\left(1 - e^{-\lambda}\right)^{\cancel{2}}} = \frac{1}{1 - e^{-\lambda}}
\end{aligned}
$$

Therefore, since the expectation of a Geometric distribution on $\{1, 2, \dots, \}$ is simply 1 divided by the parameter p, this seems to indicate that $p = 1 - e^{-\lambda}$. One can use this fact to guide the factorization of $\mathbb{P}(Y = y)$ into the more standard form of the p.m.f. of a Geometric distribution on $\{1, 2, \dots\}$.

Extra Problems

Problem 3: Rounding

The true concentration of radiation in a particular room (measured in counts per second) is uniformly distributed on the interval $[0, 10]$. A Geiger counter is used to measure the radiation in this room, however it is very crude and only displays measurements rounded to the nearest integer value. Let X denote the true amount of radiation in the room, and Y denote the amount of radiation displayed on the Geiger counter.

(a) Is X discrete or continuous? What about Y?
(b) Is it correct to say that Y is uniformly distributed on S_Y, the state space of Y?
(c) Now, find the p.m.f. of Y.

Solution:

(a) X is continuous, whereas Y is discrete. Specifically, $S_X = [0, 10]$ whereas $Y = \{0, 1, 2, \cdots, 10\}$.

(b) No, it is not correct: there are more points in S_X that get mapped to 1 than 0.

(c) We try to relate the p.m.f. of Y to the c.d.f. of X:

$$p_Y(y) := \mathbb{P}(Y = y) = \mathbb{P}(\text{round}(X) = y) = \mathbb{P}(y - 0.5 \leq X < y + 0.5)$$

That is,

$$p_Y(Y) = F_X(y + 0.5) - F_X(y - 0.5)$$

Now, recall the c.d.f. of X takes the form

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{x}{10} & \text{if } 0 \leq x \leq 10 \\ 1 & \text{if } x \geq 10 \end{cases}$$

Therefore:

$$\begin{aligned} p_Y(Y) &= F_X(y + 0.5) - F_X(y - 0.5) \\ &= \begin{cases} 0 & \text{if } y + 0.5 \leq 0 \\ \frac{y+0.5}{10} & \text{if } 0 \leq y + 0.5 \leq 10 \\ 1 & \text{if } y + 0.5 \geq 10 \end{cases} - \begin{cases} 0 & \text{if } y - 0.5 \leq 0 \\ \frac{y-0.5}{10} & \text{if } 0 \leq y - 0.5 \leq 10 \\ 1 & \text{if } y - 0.5 \geq 10 \end{cases} \\ &= \begin{cases} 0 & \text{if } y \leq -0.5 \\ \frac{y+0.5}{10} & \text{if } -0.5 \leq y \leq 9.5 \\ 1 & \text{if } y \geq 9.5 \end{cases} - \begin{cases} 0 & \text{if } y \leq 0.5 \\ \frac{y-0.5}{10} & \text{if } 0.5 \leq y \leq 10.5 \\ 1 & \text{if } y \geq 10.5 \end{cases} \\ &= \begin{cases} 0 & \text{if } y \leq -0.5 \\ \frac{y+0.5}{10} & \text{if } -0.5 \leq y \leq 0.5 \\ \frac{y+0.5}{10} & \text{if } 0.5 \leq y \leq 9.5 \\ 1 & \text{if } 9.5 \leq y \leq 10.5 \\ 1 & \text{if } y \geq 10.5 \end{cases} - \begin{cases} 0 & \text{if } y \leq -0.5 \\ 0 & \text{if } -0.5 \leq y \leq 0.5 \\ \frac{y-0.5}{10} & \text{if } 0.5 \leq y \leq 9.5 \\ \frac{y-0.5}{10} & \text{if } 9.5 \leq y \leq 10.5 \\ 1 & \text{if } y \geq 10.5 \end{cases} \end{aligned}$$

$$= \begin{cases} 0 - 0 & \text{if } y \leq -0.5 \\ \frac{y+0.5}{10} - 0 & \text{if } -0.5 \leq y \leq 0.5 \\ \frac{y+0.5}{10} - \frac{y-0.5}{10} & \text{if } 0.5 \leq y \leq 9.5 \\ 1 - \frac{y-0.5}{10} & \text{if } 9.5 \leq y \leq 10.5 \\ 1 - 1 & \text{if } y \geq 10.5 \end{cases}$$

$$= \begin{cases} 0 & \text{if } y \leq -0.5 \\ \frac{y+0.5}{10} & \text{if } -0.5 \leq y \leq 0.5 \\ \frac{1}{10} & \text{if } 0.5 \leq y \leq 9.5 \\ \frac{10.5-y}{10} & \text{if } 9.5 \leq y \leq 10.5 \\ 0 & \text{if } y \geq 10.5 \end{cases}$$

First, we combine the $y \leq -0.5$ and $y \geq 10.5$ into a single "otherwise" category:

$$p_Y(y) = \begin{cases} \frac{y+0.5}{10} & \text{if } -0.5 \leq y \leq 0.5 \\ \frac{1}{10} & \text{if } 0.5 \leq y \leq 9.5 \\ \frac{10.5-y}{10} & \text{if } 9.5 \leq y \leq 10.5 \\ 0 & \text{otherwise} \end{cases}$$

Now, we note the support of Y. Notice that the only value between -0.5 and 0.5 that Y can attain is the value 0. Similarly, the only values between 0.5 and 9.5 that Y can attain are $1, 2, \dots 9$, and the only value between 9.5 and 10.5 that Y can attain is the value 10. Thus, we can convert our expression above into:

$$p_Y(y) = \begin{cases} \frac{y+0.5}{10} & \text{if } y = 0 \\ \frac{1}{10} & \text{if } y \in \{1, 2, \dots, 9\} \\ \frac{10.5-y}{10} & \text{if } y = 10 \\ 0 & \text{otherwise} \end{cases}$$

which simplifies to

$$\boxed{p_Y(y) = \begin{cases} {}^{0.5}/_{10} & \text{if } y = 0 \\ {}^{1}/_{10} & \text{if } y \in \{1, 2, \dots, 9\} \\ {}^{0.5}/_{10} & \text{if } y = 10 \\ 0 & \text{otherwise} \end{cases}}$$

As a sanity check, note that the p.m.f. values do indeed sum to unity.

Problem 4: Transformations *(CB, 2.1)*

In each of the following find the p.d.f. of Y. Show that the p.d.f. integrates to 1.

Solution: Note: Each part can be solved using either the CDF method or by utilizing the Change of Variable formula. Just so everyone gets some practice with the Change of Variable formula, though, I shall utilize that method in all parts below.

(a) $Y = X^3$ and $f_X(x) = 42x^5(1-x),\ 0 < x < 1$

Solution:

- $g(t) = t^3$ is invertible over $[0, 1]$.
- $g^{-1}(y) = \sqrt[3]{y} = y^{1/3}$
- $\left|\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y)\right| = \left|\frac{1}{3y^{2/3}}\right| = \frac{1}{3y^{2/3}}$
- $$\begin{aligned} f_X[g^{-1}(y)] = f_X(y^{1/3}) &= \begin{cases} 42\left(y^{1/3}\right)^5\left(1 - y^{1/3}\right) & \text{if } 0 < y^{1/3} < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 42y^{5/3}\left(1 - y^{1/3}\right) & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= 42y^{5/3}(1 - y^{1/3}) \cdot \mathbb{1}_{\{y \in (0,1)\}} \end{aligned}$$

  Thus, putting everything together,

  $$\begin{aligned} f_Y(y) &= 42y^{5/3}(1 - y^{1/3}) \cdot \mathbb{1}_{\{y \in (0,1)\}} \cdot \frac{1}{3y^{2/3}} \\ &= 14y(1 - y^{1/3}) \cdot \mathbb{1}_{\{y \in (0,1)\}} \\ &= 14\left(y - y^{4/3}\right) \cdot \mathbb{1}_{\{y \in (0,1)\}} \end{aligned}$$

  As a check:

  $$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y)\ \mathrm{d}y &= \int_0^1 14\left(y - y^{4/3}\right)\ \mathrm{d}y \\ &= 14\left(\frac{1}{2} - \frac{3}{7}\right) = 14 \cdot \frac{1}{14} = 1 \ \checkmark \end{aligned}$$

(b) $Y = 4X + 3$ and $f_X(x) = 7e^{-7x},\ 0 < x < \infty$

Solution:

- $g(t) = 4t + 3$ is invertible over $(0, \infty)$.
- $g^{-1}(y) = \frac{y-3}{4}$
- $\left|\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y)\right| = \left|\frac{y-3}{4}\right| = \frac{1}{4}$
- $$\begin{aligned} f_X[g^{-1}(y)] = f_X\left(\frac{y-3}{4}\right) &= \begin{cases} 7\exp\left\{-7\left(\frac{y-3}{4}\right)\right\} & \text{if } 0 < \frac{y-3}{4} < \infty \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 7\exp\left\{-7\left(\frac{y-3}{4}\right)\right\} & \text{if } y > 3 \\ 0 & \text{otherwise} \end{cases} \\ &= 7e^{-\frac{7}{4}(y-3)} \cdot \mathbb{1}_{\{y > 3\}} \end{aligned}$$

Thus, putting everything together,

$$f_Y(y) = 7e^{-\frac{7}{4}(y-3)} \cdot \mathbb{1}_{\{y>3\}} \cdot \frac{1}{4} = \frac{7}{4}e^{-\frac{7}{4}(y-3)} \cdot \mathbb{1}_{\{y>3\}}$$

As a check:

$$\int_{-\infty}^{\infty} f_Y(y)\ \mathrm{d}y = \int_3^{\infty} \frac{7}{4}e^{-\frac{7}{4}(y-3)}\ \mathrm{d}y$$

Make a u−substitution: $u = (7/4)(y-3)$ so that $\mathrm{d}u = (7/4)\ \mathrm{d}u$ and

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y)\ \mathrm{d}y &= \int_3^{\infty} \frac{7}{4}e^{-\frac{7}{4}(y-3)}\ \mathrm{d}y \\ &= \int_0^{\infty} e^{-u}\ \mathrm{d}u = -e^{-u}\Big]_{u=0}^{u=\infty} = 0-(-1) = 1 \checkmark \end{aligned}$$

(c) $Y = X^2$ and $f_X(x) = 30x^2(1-x)^2,\ 0 < x < 1$

Solution:

- $g(t) = t^2$ is invertible over $[0, 1]$.
- $g^{-1}(y) = \sqrt{y} = y^{1/2}$
- $\left|\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y)\right| = \left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{2\sqrt{y}}$
- $$\begin{aligned} f_X[g^{-1}(y)] = f_X(\sqrt{y}) &= \begin{cases} 30\left(\sqrt{y}\right)^2\left(1-\sqrt{y}\right)^2 & \text{if } 0 < \sqrt{y} < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 30y\left(1-\sqrt{y}\right)^2 & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= 30y\left(1-\sqrt{y}\right)^2 \cdot \mathbb{1}_{\{y\in(0,1)\}} \end{aligned}$$

Thus, putting everything together,

$$\begin{aligned} f_Y(y) &= 30y\left(1-\sqrt{y}\right)^2 \cdot \mathbb{1}_{\{y\in(0,1)\}} \cdot \frac{1}{2\sqrt{y}} \\ &= 15\sqrt{y}\left(1-\sqrt{y}\right)^2 \cdot \mathbb{1}_{\{y\in(0,1)\}} \end{aligned}$$

As a check:

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y)\ \mathrm{d}y &= \int_0^1 15\sqrt{y}\left(1-\sqrt{y}\right)^2\ \mathrm{d}y \\ &= 15\int_0^1 \sqrt{y}(1-2\sqrt{y}+y)\ \mathrm{d}y \\ &= 15\int_0^1 \left(y^{1/2} - 2y + y^{3/2}\right)\ \mathrm{d}y \end{aligned}$$

$$= 15 \cdot \left(\frac{2}{3} - 1 + \frac{2}{5} \right) = 15 \cdot \frac{1}{15} = 1 \checkmark$$

Problem 5: Square-y Situation

Suppose $X \sim \text{Unif}[-1, 2]$ and $Y := X^2$.

(a) Compute $\mathbb{E}[Y]$. **Hint:** If you remember certain properties about the uniform distribution, you can do this without computing any integrals.

Solution: $\mathbb{E}[Y] = \mathbb{E}[X^2] = \text{Var}(X) + [\mathbb{E}(X)]^2$. Since $X \sim \text{Unif}[-1, 2]$ we know that

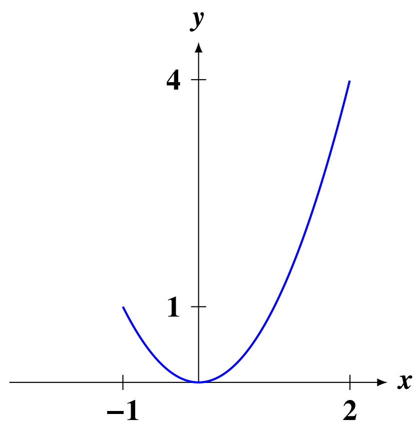
$$\mathbb{E}(X) = \frac{1}{2}; \quad \text{Var}(X) = \frac{[2 - (-1)]^2}{12} = \frac{9}{12} = \frac{3}{4}$$

and so

$$\mathbb{E}[Y] = \frac{3}{4} + \left(\frac{1}{2} \right)^2 = \frac{5}{4}$$

(b) Find $f_Y(y)$, the probability density function (p.d.f.) of Y.

Solution: Let's take a look at what $g(x) = x^2$ looks like over the state space of X:



From the $y-$axis of this graph, we can see that $S_Y = [0, 4]$. Additionally, we see that $g(x) = x^2$ fails to be invertible over the entire state space of X, meaning we must partition S_X. A natural partition is $S_X^{(1)} = [-1, 0]$ and $S_X^{(2)} = [0, 2]$.

- $\underline{S_X^{(1)}}$: We see firstly that $S_X^{(1)} \mapsto [0, 1]$. Therefore, we fix a $y \in [0, 1]$ and compute the portion of the density of Y that results: over $S_X^{(1)}$ we have $g^{-1}(y) = -\sqrt{y}$ meaning
$$\left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| = \frac{1}{2\sqrt{y}}$$
We also know that since $X \sim \text{Unif}[-1, 2]$, we have $f_X(x) = {}^1\!/_3 \cdot \mathbb{1}_{\{x \in [-1, 2]\}}$ and so, by the change of variable formula, we find
$$f_Y^{(1)}(y) = \frac{1}{3} \cdot \frac{1}{2\sqrt{y}} \cdot \mathbb{1}_{\{y \in [0, 1]\}} = \frac{1}{6\sqrt{y}} \cdot \mathbb{1}_{\{y \in [0, 1]\}}$$

- $\underline{S_X^{(2)}}$: We see firstly that $S_X^{(2)} \mapsto [0,4]$. Therefore, we fix a $y \in [0,4]$ and compute the portion of the density of Y that results: over $S_X^{(2)}$ we have $g^{-1}(y) = +\sqrt{y}$ meaning

$$\left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| = \frac{1}{2\sqrt{y}}$$

We also still have have $f_X(x) = {}^1\!/\!{}_3 \cdot \mathbb{1}_{\{x \in [-1,2]\}}$ meaning, by the change of variable formula, we have

$$f_Y^{(2)}(y) = \frac{1}{3} \cdot \frac{1}{2\sqrt{y}} \cdot \mathbb{1}_{\{y \in [0,4]\}} = \frac{1}{6\sqrt{y}} \cdot \mathbb{1}_{\{y \in [0,4]\}}$$

Our final density $f_Y(y)$ will be the sum of these two "sub-densities":

$$f_Y(y) = f_Y^{(1)}(y) + f_Y^{(2)}(y) = \frac{1}{6\sqrt{y}} \cdot \mathbb{1}_{\{y \in [0,1]\}} + \frac{1}{6\sqrt{y}} \cdot \mathbb{1}_{\{y \in [0,4]\}}$$

What we see is that for $y \in [0,1]$ *both* of the indicators above are nonzero, meaning

$$f_Y(y) = \frac{1}{6\sqrt{y}} + \frac{1}{6\sqrt{y}} = \frac{1}{3\sqrt{y}} \quad \text{if } y \in [0,1]$$

If instead $y \in [1,4]$ the leftmost indicator is zero whereas the rightmost indicator is nonzero, meaning

$$f_Y(y) = \frac{1}{6\sqrt{y}} + 0 = \frac{1}{6\sqrt{y}} \quad \text{if } y \in [1,4]$$

Finally, if $y \notin [0,4]$ both indicators are zero. Therefore,

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & \text{if } y \in [0,1] \\ \frac{1}{6\sqrt{y}} & \text{if } y \in [1,4] \\ 0 & \text{otherwise} \end{cases}}$$

As a quick check:

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y)\,\mathrm{d}y &= \int_0^1 \frac{1}{3\sqrt{y}}\,\mathrm{d}y + \int_1^4 \frac{1}{6\sqrt{y}}\,\mathrm{d}y \\ &= \frac{1}{3}\left[2\sqrt{y}\right]_{y=0}^{y=1} + \frac{1}{6}\left[2\sqrt{y}\right]_{y=1}^{y=4} = \frac{2}{3}(1-0) + \frac{1}{3}(2-1) = \frac{2}{3} + \frac{1}{3} = 1 \checkmark \end{aligned}$$