PSTAT 120A, Summer 2022: Practice Problems 6

Week 4

Conceptual Review

(a) Geometrically, what does a double integral represent?

(b) How is the evaluation of a double integral contingent on the order of integration?

(c) What is a random vector?

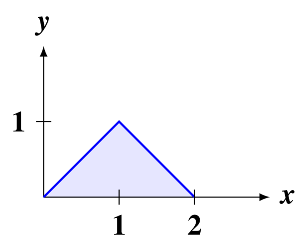
(d) What is a joint p.d.f. / joint p.m.f.?

Problem 1: Picking Points

A point is picked uniformly from the region $\mathcal{R}$ which is the area underneath the portion of the graph of $y = 1 - |x - 1|$ lying in the first quadrant. Let X denote the x−coordinate of this point and let Y denote the y−coordinate of this point.

(a) Find $f_{X,Y}(x, y)$, the joint p.d.f. of (X, Y).

Solution: Let's first sketch $\mathcal{R}$:



We can also see that

$$\text{area}(\mathcal{R}) = 2 \cdot \frac{1}{2}(1)(1) = 1$$

meaning our joint density will be

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{1} & \text{if } (x, y) \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases} = \mathbb{1}_{\{(x,y)\in\mathcal{R}\}}$$

(b) Compute $\mathbb{E}[XY]$.

Solution: Either order of integration is fine; I shall demonstrate dy dx:

$$\begin{aligned} \mathbb{E}[XY] &= \int_0^1 \int_0^x xy \, \mathrm{d}x \, \mathrm{d}y + \int_1^2 \int_0^{2-x} xy \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 x \cdot \frac{1}{2}x^2 \, \mathrm{d}x + \int_1^2 x \cdot \frac{1}{2}(2-x)^2 \, \mathrm{d}x \\ &= \frac{1}{8} + \frac{1}{2} \int_1^2 (4x - 4x^2 + x^3) \, \mathrm{d}x \end{aligned}$$

$$= \frac{1}{8} + \frac{1}{2} \cdot \frac{5}{12} = \frac{1}{3}$$

(c) Find $f_X(x)$ and $f_Y(y)$, the marginal densities of X and Y respectively.

Solution: Let's focus on $f_X(x)$ first. We will need to split into two cases:

- If $x \in [0, 1]$ we have

$$f_X(x) = \int_0^x (1)\ \mathrm{d}y = x$$

- If $x \in [1, 2]$, we have

$$f_X(x) = \int_0^{2-x} (1)\ \mathrm{d}y = 2 - x$$

Therefore, putting everything together,

$$f_X(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

For $f_Y(y)$ we play a similar game, except we now have only one case to consider: for $y \in [0, 1]$ we have

$$f_Y(y) = \int_y^{2-y} (1)\ \mathrm{d}x = 2 - 2y = 2(1 - y)$$

meaning

$$f_Y(y) = \begin{cases} 2(1 - y) & \text{if } y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Problem 2: Dice Dice Baby

Suppose Xavier and Yolanda each roll a fair k−sided die (where $k \in \mathbb{N}$). Let X denote the result of Xavier's roll and Y denote the result of Yolanda's roll, and set $Z := \max\{X, Y\}$.

a) Find the p.m.f. of Z.

Hint: First find the c.d.f. of Z and then consider how $F_Z(z)$ and $F_Z(z-1)$ can help us extract the value of $\mathbb{P}(Z = z)$.

Solution: We first examine the second hint. Note that, for arbitrary random variables,

$$\begin{aligned} F_Z(z - 1) &= \mathbb{P}(Z \le z - 1) \\ F_Z(z) = \mathbb{P}(Z \le z) &= \mathbb{P}(Z \le z - 1) + \mathbb{P}(Z = z) \\ &= F_Z(z) - \mathbb{P}(Z = z) \end{aligned}$$

Re-arranging terms yields the following useful result:

$$\mathbb{P}(Z = z) = F_Z(z) - F_Z(z - 1) \tag{1}$$

Now, let us derive an expression for the c.m.f. of Z; that is, we examine

$$F_Z(z) := \mathbb{P}(Z \leq z) = \mathbb{P}(\max\{X, Y\} \leq z)$$

Note that $\{\max\{X, Y\} \leq z\} \implies \{X \leq z\} \cap \{Y \leq z\}$. [See below for a short proof of this fact, if you are still a bit confused.] Therefore, we may write:

$$\begin{aligned} F_Z(z) &= \mathbb{P}(\{X \leq z\} \cap \{Y \leq y\}) \\ &= \mathbb{P}(X \leq z)\mathbb{P}(Y \leq z) = F_X(z) \cdot F_Y(z) \end{aligned}$$

where we have utilized the independence of X and Y to split the probability of the intersection into a product of two probabilities. Additionally, since X and Y follow the same distribution, they will have the same c.m.f. and therefore

$$F_Z(z) = [F_X(z)]^2$$

Since $X \sim \text{Unif}\{1, \cdots, k\}$ we have that

$$F_X(z) = \frac{z}{k}; \quad z \in \{1, \ldots, k\}$$

and so

$$F_Z(z) = \frac{z^2}{k^2} \tag{2}$$

Now, by equation (1), we find

$$\begin{aligned} \mathbb{P}(Z = z) &= F_Z(z) - F_Z(z - 1) \\ &= \frac{z^2}{k^2} - \frac{(z-1)^2}{k^2} \\ &= \frac{z^2 - (z-1)^2}{k^2} \\ &= \frac{z^2 - (z^2 - 2z + 1)}{k^2} \\ &= \frac{\cancel{z^2} - \cancel{z^2} + 2z - 1}{k^2} = \frac{2z-1}{k^2} \end{aligned}$$

This is, of course, only valid for $z \in \{1, \ldots, k\}$ meaning our full p.m.f. for Z is

$$\mathbb{P}(Z = z) = \begin{cases} \frac{2z-1}{k^2} & \text{if } z \in \{1, \ldots, k\} \\ 0 & \text{otherwise} \end{cases}$$

b) Verify that your expression from part (a) sums to 1, when summed over the appropriate values. It may be useful to recall that

$$\sum_{i=1}^{w} i = \frac{w(w+1)}{2}$$

Solution: We compute

$$\begin{aligned}
\sum_{\text{all } z} \mathbb{P}(Z = z) &= \sum_{z=1}^{k} \frac{2z-1}{k^2} \\
&= \frac{1}{k^2}\left[2\sum_{z=1}^{k} z - \sum_{z=1}^{k} 1\right] \\
&= \frac{1}{k^2}\left[\cancel{2} \cdot \frac{k(k+1)}{\cancel{2}} - k\right] \\
&= \frac{1}{k^2}(k^2 + \cancel{k} - \cancel{k}) = \frac{k^2}{k^2} = 1 \checkmark
\end{aligned}$$

c) Compute $\mathbb{E}(Z)$. It may be useful to recall that

$$\sum_{i=1}^{w} i^2 = \frac{w(w+1)(2w+1)}{6}$$

Solution: We compute

$$\begin{aligned}
\mathbb{E}(Z) &= \sum_{\text{all } z} z\mathbb{P}(Z = z) \\
&= \sum_{z=1}^{k} z \cdot \frac{2z-1}{k^2} \\
&= \frac{1}{k^2}\sum_{z=1}^{k}(2z^2 - z) \\
&= \frac{1}{k^2}\left[2\sum_{z=1}^{k} z^2 - \sum_{z=1}^{k} z\right] \\
&= \frac{1}{k^2}\left[2 \cdot \frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2}\right] \\
&= \frac{1}{k}\left[\frac{(k+1)(2k+1)}{3} - \frac{k+1}{2}\right] \\
&= \frac{1}{k}\left[\frac{2(k+1)(2k+1) - 3(k+1)}{6}\right] \\
&= \frac{(k+1)[2(2k+1) - 3]}{6k}
\end{aligned}$$

$$= \frac{(k+1)(4k+2-3)}{6k} = \frac{(k+1)(4k-1)}{6k}$$

Extra Problems

Problem 3: Continuous Joint Density

Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(x, y) = \begin{cases} c(x+y) & 0 < x < 1, \quad 1 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant.

(a) What is the value of c?

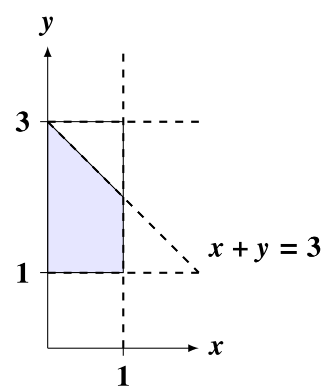
Solution:

$$\begin{aligned} 1 &= \int_1^3 \int_0^1 c(x+y)\,\mathrm{d}x\,\mathrm{d}y \\ &= c\int_1^3 \left[\frac{1}{2}x^2 + xy\right]_{x=0}^{x=1} \mathrm{d}y \\ &= c\int_1^3 \left(\frac{1}{2} + y\right) \mathrm{d}y \\ &= c\left[\frac{1}{2}(3-1) + \frac{1}{2}(9-1)\right] = 5c \end{aligned}$$

Thus, we have $c = 1/5$.

(b) Set up the integral to find $P(X + Y < 3)$. Do not evaluate.

Solution: If we draw a picture of the region, we have a rectangle with the line $y = 3 - x$ going through it. The area we want to integrate over is the area below the line and inside the rectangle.



$$\begin{aligned} P(X + Y < 3) &= P(Y < 3 - X) \\ &= \int_0^1 \int_1^{3-x} \frac{1}{5}(x+y)\,\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

Alternatively, we could have integrated with respect to *dx dy* instead (and, in that case, written two integrals).

$$P(X+Y<3) = \int_1^2 \int_0^1 \frac{1}{5}(x+y)\, \mathrm{d}x\, \mathrm{d}y + \int_2^3 \int_0^{3-y} \frac{1}{5}(x+y)\, \mathrm{d}x\, \mathrm{d}y$$

(c) Use the definition of independence to prove whether or not X and Y are independent.

Solution: Random variables X and Y are independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

We'll calculate the marginals of X and Y and see if $f_{X,Y}(x,y) = f_X(x) f_Y(y)$. For the marginal of X:

$$\begin{aligned} f_X(x) &= \int_y f_{X,Y}(x,y)\, \mathrm{d}y \\ &= \int_1^3 \frac{1}{5}(x+y)\, \mathrm{d}y \\ &= \frac{1}{5}\left[xy + \frac{1}{2}y^2\right]_{y=1}^{y=3} \\ &= \frac{1}{5}\left[3x + \frac{9}{2} - x - \frac{1}{2}\right] = \frac{1}{5}(2x+4) \end{aligned}$$

which is valid for $x \in (0,1)$, meaning

$$f_X(x) = \begin{cases} \frac{1}{5}(2x+4) & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

For the marginal of Y:

$$\begin{aligned} f_Y(y) &= \int_x f_{X,Y}(x,y)\, \mathrm{d}x \\ &= \int_0^1 \frac{1}{5}(x+y)\, \mathrm{d}x \\ &= \frac{1}{5}\left[\frac{1}{2}x^2 + xy\right]_{x=0}^{x=1} \\ &= \frac{1}{5}\left[\frac{1}{2} + y\right] = \frac{1}{10}(2y+1) \end{aligned}$$

which is valid for $y \in (1,3)$, meaning

$$f_Y(y) = \begin{cases} \frac{1}{10}(2y+1) & \text{if } y \in (1,3) \\ 0 & \text{otherwise} \end{cases}$$

Thus, we see that

$$\begin{aligned} f_X(x) \cdot f_Y(y) &= \begin{cases} \frac{1}{5}(2x+4) & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases} \cdot \begin{cases} \frac{1}{10}(2y+1) & \text{if } y \in (1,3) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{50}(2x+4)(2y+1) & \text{if } 0 < x < 1 \ , \ 1 < y < 3 \\ 0 & \text{otherwise} \end{cases} \\ & \begin{cases} \frac{1}{50}(4xy+2x+8y+4) & \text{if } 0 < x < 1 \ , \ 1 < y < 3 \\ 0 & \text{otherwise} \end{cases} \\ &\neq f_{X,Y}(x,y) \end{aligned}$$

Therefore, X and Y are NOT independent.

Problem 4: Discrete Joint Density

Suppose that X and Y are jointly distributed discrete random variables with joint probability mass function (P.M.F.) given by

$$p_{X,Y}(x,y) = \begin{cases} \binom{5}{x}\left(\frac{1}{2}\right)^{y+5} & \text{if } x \in \{0,1,2,3,4,5\} \ , \ y \in \{1,2,\cdots\} \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that $p_{X,Y}(x,y)$ is a valid joint pmf.

Solution:

$$\begin{aligned} \sum_{y=1}^{\infty}\sum_{x=0}^{x=5}\binom{5}{x}\left(\frac{1}{2}\right)^{y+5} &= \sum_{y=1}^{\infty}\left[\sum_{x=0}^{5}\binom{5}{x}\left(\frac{1}{2}\right)^{5}\right]\left(\frac{1}{2}\right)^{y} \\ &= \sum_{y=1}^{\infty} 1 \cdot \left(\frac{1}{2}\right)^{y} = 1 \ \checkmark \end{aligned}$$

(b) Identify the marginal distributions of X and Y **by name**, being sure to include any/all relevant parameter(s).

Solution: The easiest way to approach this part is to note that the joint pmf factors as

$$p_{X,Y}(x,y) = \left[\binom{5}{x}\left(\frac{1}{2}\right)^{5}\right] \cdot \left[\left(\frac{1}{2}\right)^{y}\right]$$

which shows that $X \sim \text{Bin}(5, {}^1\!/_2)$ and $Y \sim \text{Geom}({}^1\!/_2)$. Alternatively, we could have summed over x and y separately to obtain the marginal p.m.f.'s.

(c) Use your answer to part (b) to compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Solution: $\mathbb{E}[X] = 5/2$; $\mathbb{E}(Y) = 2$.

(d) Find $\text{Corr}(X, Y)$. **Hint:** based in your answer to part (b), you should not need to do any algebra.

Parts (d) and (e) involve material from Thursday's lecture.

Solution: By part (b) we see that $X \perp Y$, meaning $\text{Corr}(X, Y) = 0$.

(e) Compute $\mathbb{E}[XY]$. **Hint:** You can either perform a double summation, or you can use your answer to part (d).

Solution: In general, $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. Since $\text{Cov}(X, Y) = 0$ by part (d) above, we see that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 5/2 \cdot 2 = 5$.