PSTAT 120A, Summer 2022: Practice Problems 7

Week 5

Conceptual Review

(a) Why is the sum of two random variables also a random variable?

(b) What is the convolution formula?

(c) What is an indicator? How do indicators and expectations mesh?

Problem 1: Sum Useful Results

Prove each of the following results using the convolution formula.

(a) If $X \sim \text{Pois}(\lambda_X)$ and $Y \sim \text{Pois}(\lambda_Y)$ with $X \perp Y$, then $(X+Y) \sim \text{Pois}(\lambda_X + \lambda_Y)$.

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Solution: Since X and Y are discrete, we shall utilize the Discrete Convolution. First note that

$$\begin{aligned} p_{X,Y}(x,y) &= p_X(x) \cdot p_Y(y) \\ &= e^{-\lambda_x} \cdot \frac{\lambda_x^x}{x!} \cdot \mathbb{1}_{\{x \in \{0,1,2,\cdots\}} \cdot e^{-\lambda_y} \cdot \frac{\lambda_x^y}{y!} \cdot \mathbb{1}_{\{y \in \{0,1,2,\cdots\}} \\ p_{X,Y}(x, z-x) &= e^{-\lambda_x} \cdot \frac{\lambda_x^x}{x!} \cdot \mathbb{1}_{\{x \in \{0,1,2,\cdots\}} \cdot e^{-\lambda_y} \cdot \frac{\lambda_y^{z-x}}{(z-x)!} \cdot \mathbb{1}_{\{z-x \in \{0,1,2,\cdots\}} \end{aligned}$$

Let's focus on the product of the indicators for a minute. In order for the joint to be nonzero, we require both $x \in \{0, 1, 2\cdots\}$ and $z - x \in \{0, 1, 2, \cdots\}$. This second condition implies $x \in \{z,\ z-1,\ z-2, \cdots\}$ which, when combined with the first condition, requires $x \in \{0, 1, \cdots, z\}$. For an $x \in \{0, 1, \cdots, z\}$ we have

$$\begin{aligned} p_{X,Y}(x,y) &= e^{-\lambda_x} \cdot \frac{\lambda_x^x}{x!} \cdot e^{-\lambda_y} \cdot \frac{\lambda_y^{z-x}}{(z-x)!} \\ &= e^{-(\lambda_x + \lambda_y)} \cdot \frac{1}{x!(z-x)!} \cdot \lambda_y^z \cdot \left(\frac{\lambda_x}{\lambda_y}\right)^x \end{aligned}$$

and so

$$\begin{aligned} p_Z(z) &= \sum_x p_{X,Y}(x, z-x) \\ &= \sum_{x=0}^{z} e^{-(\lambda_x+\lambda_y)} \cdot \frac{1}{x!(z-x)!} \cdot \lambda_y^z \cdot \left(\frac{\lambda_x}{\lambda_y}\right)^x \\ &= e^{-(\lambda_x+\lambda_y)} \cdot \lambda_y^z \cdot \sum_{x=0}^{z} \frac{1}{x! \cdot (z-x)!} \left(\frac{\lambda_x}{\lambda_y}\right)^x \\ &= e^{-(\lambda_x+\lambda_y)} \cdot \lambda_y^z \cdot \frac{1}{z!} \sum_{x=0}^{z} \frac{z!}{x! \cdot (z-x)!} \left(\frac{\lambda_x}{\lambda_y}\right)^x \cdot (1)^{z-x} \\ &= e^{-(\lambda_x+\lambda_y)} \cdot \lambda_y^z \cdot \frac{1}{z!} \sum_{x=0}^{z} \binom{z}{x} \left(\frac{\lambda_x}{\lambda_y}\right)^x (1)^{z-x} \end{aligned}$$

$$= e^{-(\lambda_x+\lambda_y)} \cdot \lambda_y^z \cdot \frac{1}{z!} \cdot \left(1 + \frac{\lambda_x}{\lambda_y}\right)^z$$

$$= e^{-(\lambda_x+\lambda_y)} \cdot \frac{1}{z!} \cdot \cancel{\lambda_y^z} \left(\frac{\lambda_x+\lambda_y}{\cancel{\lambda_y}}\right)^z$$

$$= e^{-(\lambda_x+\lambda_y)} \cdot \frac{(\lambda_x+\lambda_y)^z}{z!}$$

The state space of Z is clearly $S_Z = \{0, 1, \cdots\}$ meaning, in conjunction with the p.m.f. found above, $Z \sim \text{Pois}(\lambda_x + \lambda_y)$

(b) If $X \sim \text{Gamma}(r_x, \lambda)$ and $Y \sim \text{Gamma}(r_y, \lambda)$ with $X \perp Y$, then $(X+Y) \sim \text{Gamma}(r_X + r_Y,\ \lambda)$.

*Hint: You will need to use the so-called **Beta Integral**:*

$$\int_0^1 x^{r-1}(1-x)^{s-1}\,\mathrm{d}x = \frac{\Gamma(r)\cdot\Gamma(s)}{\Gamma(r+s)}$$

Solution: We note

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$$

$$= \frac{\lambda^{r_x}}{\Gamma(r_x)} \cdot x^{r_x-1} \cdot e^{-\lambda x} \cdot \mathbb{1}_{\{x\geq 0\}} \cdot \frac{\lambda^{r_y}}{\Gamma(r_y)} \cdot y^{r_y-1} \cdot e^{-\lambda y} \cdot \mathbb{1}_{\{y\geq 0\}}$$

$$= \frac{\lambda^{r_x+r_y}}{\Gamma(r_x)\cdot\Gamma(r_y)} \cdot x^{r_x-1} \cdot y^{r_y-1} \cdot e^{-\lambda(x+y)} \cdot \mathbb{1}_{\{x\geq 0,\ y\geq 0\}}$$

$$f_{X,Y}(x, z-x) = \frac{\lambda^{r_x+r_y}}{\Gamma(r_x)\cdot\Gamma(r_y)} \cdot x^{r_x-1} \cdot (z-x)^{r_y-1} \cdot e^{-\lambda(\cancel{x}+z-\cancel{x})} \cdot \mathbb{1}_{\{x\geq 0,\ z-x\geq 0\}}$$

$$= \frac{\lambda^{r_x+r_y}}{\Gamma(r_x)\cdot\Gamma(r_y)} \cdot x^{r_x-1} \cdot (z-x)^{r_y-1} \cdot e^{-\lambda z} \cdot \mathbb{1}_{\{0\leq x\leq z\}}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,\mathrm{d}x$$

$$= \int_0^z \frac{\lambda^{r_x+r_y}}{\Gamma(r_x)\cdot\Gamma(r_y)} \cdot x^{r_x-1} \cdot (z-x)^{r_y-1} \cdot e^{-\lambda z}\,\mathrm{d}x$$

$$= \frac{\lambda^{r_x+r_y}}{\Gamma(r_x)\cdot\Gamma(r_y)} \cdot e^{-\lambda z} \cdot \int_0^z x^{r_x-1} \cdot (z-x)^{r_y-1}\,\mathrm{d}x$$

To evaluate this integral, we would like to use the hint. As such, let's substitute u such that $x = zu$; i.e. $u = (x/z)$ and so $\mathrm{d}u = (1/z)\,\mathrm{d}z$:

$$\int_0^z x^{r_x-1} \cdot (z-x)^{r_y-1}\,\mathrm{d}x = \int_0^1 (zu)^{r_x-1}\, z^{r_y-1} \cdot (1-u)^{r_y-1} \cdot z\,\mathrm{d}u$$

$$= z^{r_x+r_y-1} \cdot \int_0^1 u^{r_x-1}(1-u)^{r_y-1)}\,\mathrm{d}u$$

$$= z^{r_x+r_y-1} \cdot \frac{\Gamma(r_x)\cdot\Gamma(r_y)}{\Gamma(r_x+r_y)}$$

Therefore,

$$f_Z(z) = \frac{\lambda^{r_x+r_y}}{\Gamma(r_x)\cdot\Gamma(r_y)} \cdot e^{-\lambda z} \cdot \int_0^z x^{r_x-1} \cdot (z-x)^{r_y-1}\,\mathrm{d}x$$

$$\frac{\lambda^{r_x+r_y}}{\cancel{\Gamma(r_x)\cdot\Gamma(r_y)}}\cdot e^{-\lambda z}\cdot z^{r_x+r_y-1}\cdot\frac{\cancel{\Gamma(r_x)\cdot\Gamma(r_y)}}{\Gamma(r_x+r_y)}$$
$$\frac{\lambda^{r_x+r_y}}{\Gamma(r_x+r_y)}\cdot z^{r_x+r_y-1}\cdot e^{-\lambda z}$$

which, since $S_Z=[0,\infty)$, allows us to conclude that $Z\sim\text{Gamma}(r_x+r_y,\ \lambda)$.

Problem 2: The Elevator Problem

Suppose that, in a particular 10-story building, 5 people enter an elevator on the Ground Floor (let us call this "Floor 0"). We assume that people get off the elevator at a random floor, independently of all other people in the elevator. (Assume that nobody leaves on the Ground Floor) In this problem, we shall work toward answering the question: what is the expected number of floors at which the elevator will stop?

a) Let X denote the number of floors at which the elevator will stop. Define appropriate indicators $\mathbb{1}_j$ such that X can be expressed as a sum of these indicators.

Hint: We can assign indicators to people, or assign them to floors. Which will be better?

Solution: We take $\mathbb{1}_j=\begin{cases}1 & \text{if the elevator stops at floor } j\\ 0 & \text{otherwise}\end{cases}$ for $j=1,2,\dots,10$. In this way,

$$X=\sum_{j=1}^{10}\mathbb{1}_j$$

b) Using your expression from part (a), write $\mathbb{E}(X)$ in terms of $\mathbb{E}(\mathbb{1}_1),\cdots,\mathbb{E}(\mathbb{1}_{10})$. (You don't need to find the expectation of the indicators just yet; you'll do that in the next part.)

Solution: By the linearity of expectation, $\mathbb{E}(X)=\mathbb{E}(\sum_{j=1}^{10}\mathbb{1}_j)=\sum_{j=1}^{10}\mathbb{E}(\mathbb{1}_j)$.

c) Now, compute $\mathbb{E}(\mathbb{1}_1),\mathbb{E}(1_2),\dots,\mathbb{E}(1_{10})$, and use this to answer the original question of "what is the expected number of floors at which the elevator will stop?"

Hint: Using symmetry, you can find an expression for $\mathbb{E}(\mathbb{1}_j)$ for an arbitrary $j=1,2,\dots,10$.

Solution: Recall that, for the indicator $\mathbb{1}_A$ of the event A, $\mathbb{E}(\mathbb{1}_A)=\mathbb{P}(A)$. Therefore,

$$\begin{aligned}\mathbb{E}(\mathbb{1}_j)&=\mathbb{P}(\text{elevator stops at floor } j)\\&=1-\mathbb{P}(\text{elevator does not stop at floor } j)\\&=1-\mathbb{P}(\text{nobody gets off at floor } j)\end{aligned}$$

$$= 1 - \mathbb{P}(\text{all 5 people get off at one of the other 9 floors})$$
$$= 1 - \left(\frac{9}{10}\right)^5$$

Therefore, we see

$$\mathbb{E}(X) = \sum_{j=1}^{10} \mathbb{E}(\mathbb{1}_j) = \sum_{j=1}^{10} \left[1 - \left(\frac{9}{10}\right)^5\right] = 10 \cdot \left[1 - \left(\frac{9}{10}\right)^5\right] \approx 4.0951$$

Key Takeaway: This problem (hopefully) illustrates one of the many reasons why indicators are very useful, especially in the context of expectations. One can extend this logic to actually compute the *variance* of the number of floors at which the elevator will stop!

Problem 3: Poisson Predictions

Suppose that the number of calls arriving at a call center follows a Poisson Process with an average of 10 calls per hour.

(a) What is the probability that exactly 20 calls arrive in a 90-minute interval?

> **Solution:** Let X denote the number of calls in a 90-minute interval; since 90 minutes = (3/2) hours, we know that X follows a Poisson distribution with rate
>
> $$\lambda_X = 10 \cdot \frac{3}{2} = 15$$
>
> Therefore,
>
> $$\mathbb{P}(X = 20) = e^{-15} \cdot \frac{15^{20}}{20!} \approx 0.04181$$

(b) What is the probability that the 2^{nd} and 4^{th} calls arrive within 1 hour of each other?

> **Solution:** Let T denote the time, in hours, between the 2^{nd} and 4^{th} calls. We know then that $T \sim \text{Gamma}(2, 10)$ meaning
>
> $$\begin{aligned} \mathbb{P}(T < 1) &= \int_0^1 \frac{10^2}{\Gamma(2)} t e^{-10t} \, \mathrm{d}t \\ &= 100 \cdot \left[-\frac{1}{100} e^{-10t} (10t + 1) \right]_{t=0}^{t=1} = 1 - 11e^{-10} \approx 0.9995 \end{aligned}$$

(c) What is the distribution of the amount of time between the 2^{nd} and 3^{rd} calls **as measured in minutes**?

> **Solution:** If we let S denote the time in hours between the 2^{nd} and 3^{rd} calls, then we know that $S \sim \text{Exp}(10)$. Additionally, we know that 1 hour = 60 minutes; thus, if M measures the time in minutes between the 2^{nd} and 3^{rd} calls then M is a transformation of S defined by way of
>
> $$M = 60S$$
>
> We know that if $X \sim \text{Exp}(\lambda)$ then $(cX) \sim \text{Exp}(\lambda/c)$; hence
>
> $$M \sim \text{Exp}(1/6)$$
>
> As a quick sanity check: we know that the average time between the 2^{nd} and 3^{rd} calls is (1/10) of an hour; we can see that in fact (1/10) of an hour is 6 minutes, which is equal to the expectation of M.

(d) If T_1 measures the time in minutes until the 1^{st} call and S denotes the time in minutes between the 2^{nd} and 4^{th} calls, what is $f_{T_1,S}(t, s)$, the joint p.d.f. of (T_1, S)?

Solution: By a similar reasoning as we used in part (c),

$$T_1 \sim \text{Exp}(1/6)$$
$$S \sim \text{Gamma}(2, 1/6)$$

We also know that $T_1 \perp S$, meaning

$$\begin{aligned} f_{T_1,S}(t,s) &= f_{T_1}(t) \cdot f_S(s) \\ &= \left(\frac{1}{6}\right) \cdot e^{-\frac{1}{6}t} \cdot \mathbb{1}_{\{t \geq 0\}} \cdot \frac{\left(\frac{1}{6}\right)^2}{\Gamma(2)} \cdot s^{2-1} \cdot e^{-\frac{1}{6}s} \cdot \mathbb{1}_{\{s \geq 0\}} \\ &= \left(\frac{1}{6}\right)^3 s e^{-\frac{1}{6}(t+s)} \cdot \mathbb{1}_{\{t \geq 0,\ s \geq 0\}} \end{aligned}$$

Extra Problems

Problem 4: Great Expectations

Now that we have learned a bit more about joint distributions, consider the following logic in the context of a bivariate pair (X, Y) of continuous random variables:

- On the one hand, we can integrate out y, find the marginal $f_X(x)$ of X, and then compute
$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) \, \mathrm{d}x$$
- On the other hand, we can also use the two-dimensional LOTUS with $g(x, y) = x$ to compute
$$\mathbb{E}[X] = \iint_{\mathbb{R}^2} x f_{X,Y}(x, y) \, \mathrm{d}A$$

A question I often get asked is: "which of these is correct?" The answer is, in fact- "both of them!" **Prove that these two formulations of $\mathbb{E}[X]$ are equivalent.**

It may be easier to start with the second formulation, and then show that it is equal to the first.

Solution: Start with the second formulation: in other words, begin by writing

$$\mathbb{E}[X] = \iint_{\mathbb{R}^2} x \cdot f_{X,Y}(x, y) \, \mathrm{d}A$$

In. this case it doesn't really matter which variable we integrate with respect to first; thus, let's (somewhat arbitrarily) use dy dx. That is, we write

$$\begin{aligned} \mathbb{E}[X] &= \iint_{\mathbb{R}^2} x \cdot f_{X,Y}(x, y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x, y) \, \mathrm{d}x \right) \mathrm{d}y \end{aligned}$$

$$= \int_{-\infty}^{\infty} x f_X(x) \, \mathrm{d}x$$

since $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, \mathrm{d}y$.

Problem 5: Hot Cross Moments

Given an $n-$dimensional random vector $\vec{X}$, we define the $\boldsymbol{k_1, \cdots, k_n}^{\textbf{th}}$ **cross-moment** (sometimes called a **mixed-moment**) of $\vec{X}$ to be

$$\mu_{k_1,\cdots,k_n}(\vec{X}) := \mathbb{E}\left[\prod_{i=1}^{n} X_i^{k_i}\right] = \mathbb{E}\left[X_1^{k_1} \times X_2^{k_2} \times \cdots \times X_n^{k_n}\right]$$

For example, the $(3, 5)$ cross moment of a bivariate random vector is

$$\mu_{3,5}(\vec{X}) = \mathbb{E}\left[X_1^3 \cdot X_2^5\right]$$

(a) Suppose the elements of an $n-$dimensional random vector $\vec{X}$ are independent. Additionally, let $\mu_{k_i}(X_i) := \mathbb{E}[X_i^{k_i}]$ denote the $k_i{}^{\text{th}}$ moment of X_i. Derive a relationship between $\mu_{k_1,\cdots,k_n}(\vec{X})$ and the $\mu_{k_i}(X_i)'s$.

Solution: Crucially, we know that functions of independent random variables are independent. Since the X_i's are independent [by assumption], we have that the random variables $\{(X_i)^{k_i}\}_{i=1}^{n}$ are also independent. This allows us to write the expectation of their product as the product of their expectations; hence,

$$\begin{aligned} \mu_{k_1,\cdots,k_n}(\vec{X}) &:= \mathbb{E}\left[\prod_{i=1}^{n} X_i^{k_i}\right] = \mathbb{E}\left[X_1^{k_1} \times X_2^{k_2} \times \cdots \times X_n^{k_n}\right] \\ &= \prod_{i=1}^{n} \mathbb{E}[X_i^{k_i}] = \prod_{i=1}^{n} \mu_{k_i}(X_i) \end{aligned}$$

(b) Is it true that for two $n-$dimensional random vectors $\vec{X}$ and $\vec{Y}$

$$\mu_{k_1,\cdots,k_n}(\vec{X} + \vec{Y}) = \mu_{k_1,\cdots,k_n}(\vec{X}) = +\mu_{k_1,\cdots,k_n}(\vec{Y})$$

If so, provide a brief proof. If not, explain why not.

Solution: We write:

$$\mu_{k_1,\cdots,k_n}(\vec{X} + \vec{Y}) := \mathbb{E}\left[\prod_{i=1}^{n} (X_i + Y_i)^{k_i}\right] = \mathbb{E}\left[(X_1 + Y_1)^{k_1} \times (X_2 + Y_2)^{k_2} \times \cdots \times (X_n + Y_n)^{k_n}\right]$$

In general, $(X_i + Y_i)^{k_i} \neq X_i^{k_i} + Y_i^{k_i}$. Hence, the provided statement is **false**.