PSTAT 120A, Summer 2022: Practice Problems 8

Week 5

Conceptual Review

(a) What is the MGF?

(b) Where does the name "MGF" come from?

(c) What is meant by the statement, "MGF's uniquely determine distributions?"

Problem 1: The Laplace Distribution

The **Laplace Distribution** (sometimes called the Double Exponential Distribution) has probability density function

$$f_X(x) = \frac{1}{2}e^{-|x|}; \qquad x \in (-\infty, \infty)$$

(a) Verify that $f_X(x)$ is a valid p.d.f..

Solution: $\int_{-\infty}^{\infty} \frac{1}{2}e^{-|x|}\,\mathrm{d}x = \int_0^{\infty} e^{-x}\,\mathrm{d}x = 1$ ✓

(b) Find $M_X(t)$, the moment-generating function of X where X follows the Laplace distribution. Be sure to specify bounds on your expression.

Solution:

$$\begin{aligned}
M_X(t) := \mathbb{E}[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{2}e^{-|x|}\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \exp\{tx - |x|\}\,\mathrm{d}x \\
&= \frac{1}{2}\left(\int_{-\infty}^{0} \exp\{tx + x\}\,\mathrm{d}x + \int_0^{\infty} \exp\{tx - x\}\,\mathrm{d}x\right) \\
&= \frac{1}{2}\left(\int_{-\infty}^{0} e^{x(1+t)}\,\mathrm{d}x + \int_0^{\infty} e^{-x(1-t)}\,\mathrm{d}x\right)
\end{aligned}$$

For the first integral to converge, we require $t + 1 > 0$ meaning $t > -1$. For the second integral to converge we require $1 - t > 0$, or, equivalently, $t < 1$. Thus, putting these together we see that $M_X(t) < \infty$ only when $-1 < t < 1$. Thus, we henceforth assume $t \in (-1, 1)$:

$$\begin{aligned}
M_X(t) &= \frac{1}{2}\left(\int_{-\infty}^{0} e^{x(1+t)}\,\mathrm{d}x + \int_0^{\infty} e^{-x(1-t)}\,\mathrm{d}x\right) \\
&= \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right) \\
&= \frac{1}{2} \cdot \frac{1+t+1-t}{(1+t)(1-t)} = \frac{1}{1-t^2}
\end{aligned}$$

Thus, our final form for the MGF of X is

$$M_X(t) = \begin{cases} \frac{1}{1-t^2} & \text{if } |t| < 1 \\ \infty & \text{otherwise} \end{cases}$$

Now, in parts (c) through (e) we shall work toward identifying a closed-form expression for the n^{th} moment of the Laplace distribution. For notational convenience, let X be a random variable that follows the Laplace distribution.

(c) What should the value of $\mathbb{E}[X^n]$ be where n is odd? *Hint: Use symmetry*

Solution: The function $g(t) = t^n$ is odd whenever n is odd. The PDF $f_X(x)$ of X is even; hence $t^n f_X(x)$ will be odd whenever n is odd. Since $M_X(t)$ involves an integral over an interval symmetric about the origin, $\int_{-\infty}^{\infty} t^n f_X(t)\ \mathrm{d}t$ is 0 whenever n is odd; i.e. $\mathbb{E}[X^n] = 0$ whenever n is odd.

(d) Write out the MacLaurin Series Expansion of $M_X(t)$, as an infinite sum.

Solution: Fix $t \in (-1, 1)$. Then:

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$$
$$\frac{1}{1-t^2} = \sum_{k=0}^{\infty} (t^2)^k = t^{2k} = M_X(t)$$

(e) Recall that

$$M_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{\mathbb{E}[X^k]}{k!} \cdot t^k$$

Match terms from your expression in part (d) to the summation above to extract a formula for $\mathbb{E}[X^n]$ when n is even.

Solution: We have two equivalent formulations for the MGF of X; thus, we may set them equal:

$$\sum_{k=0}^{\infty} t^{2k} = \sum_{k=0}^{\infty} \frac{\mathbb{E}[X^k]}{k!} \cdot t^k$$

Note that we can write $t^{2k} = t^k \cdot \mathbb{1}_{\{k \text{ is even}\}}$, so we have

$$\sum_{k=0}^{\infty} t^k \cdot \mathbb{1}_{\{k \text{ is even}\}} = \sum_{k=0}^{\infty} \frac{\mathbb{E}[X^k]}{k!} \cdot t^k$$

meaning, matching terms,

$$\mathbb{1}_{\{k \text{ is even}\}} = \frac{\mathbb{E}[X^k]}{k!}$$

or, equivalently,

$$\mathbb{E}[X^k] = k! \cdot \mathbb{1}_{\{k \text{ is even}\}}$$

(f) It can be shown (through direct integration) that $\mathbb{E}[X^2] = 2$. Check that your answer in part (e) agrees with this fact.

Solution: $\mathbb{E}[X^2] = (2!) = 2$ ✓

Extra Problems

Problem 2: Sum More Useful Results

Prove each of the following results. You may use either the convolution formula, or MGF's.

(a) If $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, then $(X+Y) \sim \mathcal{N}(0,2)$.

(b) If $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$ with $X \perp Y$, then $(X+Y) \sim \text{Bin}(n_1 + n_2, p)$

Problem 3: More MGF's

Derive an expression for the MGF for each of the following distributions. (Yes, the final answers are in the Lecture Slides, but we're expecting you to derive them from scratch on this question!)

(a) $X \sim \text{Exp}(\lambda)$

(b) $X \sim \text{Unif}[a,b]$

(c) $Z \sim \mathcal{N}(0,1)$

(d) $X \sim \mathcal{N}(\mu, \sigma^2)$.

Hint: Instead of performing a direct integration, try and relate X to Z from part (c) via a transformation and utilize properties of MGF's.

Problem 4: A Finite Distribution

Suppose X is a discrete random variable that has moment-generating function (MGF)

$$M_X(t) = \left(\frac{2}{5}\right) e^{-4.2t} + \left(\frac{1}{5}\right) e^{t} + \left(\frac{2}{5}\right) e^{3.7t}; \qquad t \in (-\infty, \infty)$$

(a) Compute $\mathbb{E}[X]$.

Solution:

$$M_X'(t) = (4.2)\left(\frac{2}{5}\right) e^{-4.2t} + (1)\left(\frac{1}{5}\right) e^{t} + (3.7)\left(\frac{2}{5}\right) e^{3.7t}$$

$$M_X'(0) = (4.2)\left(\frac{2}{5}\right)(1)\left(\frac{1}{5}\right) + (3.7)\left(\frac{2}{5}\right) = \boxed{\frac{227}{125} = 1.816}$$

(b) Find the probability mass function (p.m.f.) of X.

Solution: We know

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k \in S_X} e^{kt}\,\mathbb{P}(X = k)$$

Suppose the elements in S_X are denoted by $s_1, s_2, \dots$; then

$$M_X(t) = e^{s_1 t}\,\mathbb{P}(X = s_1) + e^{s_2 t}\,\mathbb{P}(X = s_2) + \cdots$$

Thus, matching terms with the provided expression for the MGF yields:

k	-4.2	1	3.7
$\mathbb{P}(X = k)$	$2/5$	$1/5$	$2/5$

(c) Suppose $Y = 0.5X + 2$. Find the MGF of Y.

Solution:

$$\begin{aligned} M_Y(t) = M_{0.5X+2}(t) = \mathbb{E}[e^{(0.5X+2)t}] = \mathbb{E}[e^{(0.5t)X} \cdot e^{2t}] &= e^{2t} M_X(0.5t) \\ &= e^{2t} \left[\left(\frac{2}{5} \right) e^{-4.2\cdot(0.5t)} + \left(\frac{1}{5} \right) e^{(0.5t)} + \left(\frac{2}{5} \right) e^{3.7\cdot(0.5t)} \right] \\ &= e^{2t} \left[\left(\frac{2}{5} \right) e^{-2.1t} + \left(\frac{1}{5} \right) e^{0.5t} + \left(\frac{2}{5} \right) e^{1.85t} \right] \\ &= \left(\frac{2}{5} \right) e^{0.1t} + \left(\frac{1}{5} \right) e^{2.5t} + \left(\frac{2}{5} \right) e^{3.85t}; \quad t \in (-\infty, \infty) \end{aligned}$$

(d) Use your answer from part (c) to find the PMF of Y.

Solution: By a similar method we utilized in part (b), we find

k	0.1	2.5	3.85
$\mathbb{P}(Y = k)$	$2/5$	$1/5$	$2/5$