PSTAT 120A, Summer 2022: Practice Problems 2

Week 2

Conceptual Review

(a) Intuitively, what does $\mathbb{P}(A \mid B)$ represent?

(b) What is the definition of independence? What is the intuition behind this definition?

(c) Does pairwise independence imply mutual independence? Does mutual independence imply pairwise independence?

(d) What type of mathematical object is a Random Variable?

Problem 1: Proving Independence

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two events $A, B \in \mathcal{F}$. Show that if $A \perp B$, then $A^{\complement} \perp B^{\complement}$.

Problem 2: Conditional Complements *(modified from ASV 2.7)*

a) Argue that $\{A^{\complement} \cap B\ ,\ A \cap B\}$ forms a partition of the event B.

Hint: You can either use mathematical arguments, or sketch a Venn Diagram.

b) Show that $\mathbb{P}(A^{\complement} \mid B) = 1 - \mathbb{P}(A \mid B)$.

c) Suppose $\mathbb{P}(A \mid B) = 0.6$ and $\mathbb{P}(B) = 0.5$. Find $\mathbb{P}(A^{\complement} \cap B)$.

d) Suppose now that $A \subseteq B$. Find a simple formula for $\mathbb{P}(A \mid B^{\complement})$.

Problem 3: Selecting Words

A word is selected at random from the sentence

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Then, a letter is selected at random from the chosen word.

a) What is the probability that the letter "S" is selected?

b) If X denotes the length of the chosen word, what is the PMF of X?

c) Continuing from part (b); what is $\mathbb{E}[X]$?

d) Continuing from part (b); what is $\mathbb{E}\left[\frac{1}{X}\right]$?

e) For every vowel in your selected word, you are awarded \$1; for every consonant, however, you are forced to pay \$1. Letting W denote your net gain/loss, what is $\mathbb{E}[W]$?

Hint: Try finding the PMF of W first.

Extra Problems

Problem 4: Pólya's Urn Scheme

A box contains n marbles, b of which are blue and $g := n - b$ of which are gold. A marble is drawn at random and its color is noted; the marble is then placed back into the box along with k additional marbles of the same color (so now there are $n + k$ total marbles in the box). Now, another marble is drawn; find the probability that it is blue.

Problem 5: Number of Conditions

Recall that when establishing the independence of n events, there are a series of computations we must perform (i.e. the "two-way intersections," "three-way intersections," etc.) Show that there are a total of $2^n - n - 1$ computations involved in establishing the independence of n events.

Hint: Recall the Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Problem 6: I Like to Prove It Prove It!

Prove the following:

(a) If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Hint: Partition $A \cup B$; then note that $A \cup B = B$ since $A \subseteq B$.

(b) If $A \subseteq B$, then $\mathbb{P}(B \mid A) = 1$. Provide both a mathematical proof, as well as an intuitive one.

(c) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an event B with $\mathbb{P}(B) \neq 0$, the measure $\mathbb{P}_B(\cdot)$ defined through

$$\mathbb{P}_B(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

is a valid probability measure.

Hint: All you need to show is that $\mathbb{P}_B(\cdot)$ satisfies the three axioms of probability. Additionally, we know that $\mathbb{P}(\cdot)$ satisfies the axioms of probability.