PSTAT 120A, Summer 2022: Practice Problems 4: Midterm Review

Week 2

Conceptual Review

(a) Review the conceptual questions from the previous Discussion Worksheets!

Problem 1: Stamps!

Morgan is an avid philatelist[1], who is trying to get their hands on a very rare stamp. They know that only 2% of stores in their hometown sell this stamp, so they go from store to store trying to find this stamp. Out of desperation, Morgan will sometimes visit the same store twice.

(a) On average, how many stores (including the final store) does Morgan need to visit in order to collect 3 copies of this rare stamp?
(b) What is the probability that among the first 10 stores Morgan visits none of them have the rare stamp?

Problem 2: Poisson Predicament

In *GauchoVille*, the number of earthquakes follows a Poisson Process with rate 2 per year. Meteorologists are interested in tracking the number of earthquakes over time.

(a) What is the probability that *GauchoVille* will experience exactly 4 earthquakes in the next two years?
(b) What is the average length of time that elapses between the 3$^{\text{rd}}$ and 5$^{\text{th}}$ earthquakes?

[1] A philatelist is someone who collects stamps

Problem 3: Probabilistic Perimeters

A circle of radius R is to be constructed, where R is a random variable with probability density function (p.d.f.) given by

$$f_R(r) = \begin{cases} \frac{3}{r^4} & \text{if } r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that $f_R(r)$ is a valid p.d.f.

(b) Compute $\mathbb{P}(R \geq 3 \mid R \geq 2)$

(c) Find $F_R(r)$, the c.d.f. of R.

(d) Letting P denote the perimeter of the circle, find $F_P(p)$, the c.d.f. of P.

(e) Use your answer to part (d) to find $f_P(p)$, the p.d.f. of P.

Recall: Perimeter is 2π times the radius

Problem 4: Dicey Situation

Consider the experiment of tossing two far 6-sided dice.

(a) Write down a possible outcome space Ω. Be very clear and explicit about your notation!

(b) What is the probability that the first die lands on a number strictly less than the second die?

Problem 5: Another Dicey Situation *(Pitman, 2.Rev.18)*

Seven dice are rolled. Write down unsimplified expressions for the probabilities of each of the following events:

(a) exactly three sixes;

(b) three of one kind and four of another;

(c) two fours, two fives, and three sixes;

(d) each number appears;

(e) the sum of the dice is 9 or more.

Problem 6: Counting Cards (Again) *(Modified from ASV, 2.8)*

We shuffle a deck of cards and deal three cards (without replacement). Find the probability that the first card is a queen, the second is a king and the third is an ace. **Use methods involving Conditional Probabilities,** as opposed to simply counting favorable outcomes.

Problem 7: Prove It!

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two events $A, B \in \mathcal{F}$ with $A \perp B$. Prove that $A^{\complement} \perp B$.

Problem 8: Crafting with Cosines

Let X be a continuous random variable with probability density function (p.d.f.) given by

$$f_X(x) = \begin{cases} c \cdot |\cos x| & \text{if } x \in (-\pi/2, \pi) \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant.

(a) Find the value of c.

(b) Derive an expression for $F_X(x)$, the cumulative distribution function (c.d.f.) of X.

Hint: Your final answer should ahve 4 cases.

Problem 9: Twins *(ASV, 2.30)*

Assume that $1/3$ of all twins are identical twins. You learn that Miranda is expecting twins, but you have no other information.

Note that Identical twins must be of the same gender, whereas Fraternal twins may or may not be of the same gender.

(a) Find the probability that Miranda will have two girls.

(b) You learn that Miranda gave birth to two girls. What is the probability that the girls are identical twins?

Explain any assumptions you make.

Problem 10: Something Fishy

The weight of a fish caught in a particular pond is well-modeled by a normal distribution with mean 6lbs and standard deviation 2lbs.

(a) What is the probability that a randomly selected fish will exceed 7lbs in weight?

(b) Zeke claims to have caught a fish at the 90^{th} percentile of weights. How much does Zeke's fish weight?

Problem 11: A Poisson Calculation

(a) Prove the identity

$$\sum_{\substack{k=0 \\ \text{even}}}^{\infty} \frac{x^k}{k!} = \cosh(x)$$

Hint: Consider the expansions of e^x and e^{-x}, and see what happens when you sum these expansions together

(b) If $X \sim \text{Pois}(\lambda)$, find $\mathbb{P}(X \text{ is even})$.

Problem 12: A Geometric Calculation

(a) If $X \sim \text{Geom}(p)$, derive a simple-closed form expression for $\mathbb{P}(X \geq k)$

(b) Use the Tail-Sum Formula (which you proved on Homework 3) to re-derive the result $\mathbb{E}[X] = 1/p$.

Problem 13: Subsets *(Modified from PL 2.9.1)*

Consider the set $A = \{1, 2, \dots, n\}$, and suppose all subsets of A are equally likely to be chosen. A subset of A is randomly chosen.

(a) How many elements are in the outcome space associated with this experiment?

(b) Compute the probability that the selected subset contains the number 1.

(c) Compute the probability that the selected subset contains both the numbers 1 and 2.

(d) Compute the probability that the selected subset contains either the number 1 or 2 (or both).

Problem 14: Makin' Money Moves

A bag contains 12 red marbles and 10 blue marbles. A friend reaches in and selects 5 balls at random, with replacement. For each red marble drawn you win \$1; for each blue marble you lose \$1. Let X denote your net winnings.

(a) Find the probability mass function (p.m.f.) of X.

(b) Suppose that instead of drawing with replacement your friend now draws 5 marbles *without* replacement. Find the modified p.m.f. of X, in this new situation.

Problem 15: Inclusion-Exclusion

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and three events $A, B, C \in \mathcal{F}$, prove that

$$\begin{aligned} \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) \\ + \mathbb{P}(A \cap B \cap C) \end{aligned}$$

This is sometimes referred to as the **Inclusion-Exclusion Principle for 3 events**.

Problem 16: Practice with Axioms

Consider three events A, B, and C such that:

- $\mathbb{P}(A) = 0.45$; $\mathbb{P}(B) = 0.55$; $\mathbb{P}(C) = 0.35$
- $\mathbb{P}(A \cap B) = 0.15$; $\mathbb{P}(A \cap C) = 0.15$; $\mathbb{P}(B \cap C) = 0.15$
- $\mathbb{P}(A \cap B \cap C) = 0.05$

Compute the following probabilities:

Hint: A Venn Diagram might be helpful here.

(a) $\mathbb{P}[A^{\complement} \cup B^{\complement} \cup C^{\complement}]$

(b) $\mathbb{P}(A \cup B)$

(c) $\mathbb{P}(B \setminus C)$

(d) $\mathbb{P}(A \cup B \cup C)$

Problem 17: A Trip to the Movies

At the cinema, 70% of moviegoers purchase either popcorn or a drink (or both). Additionally: 50% of moviegoers purchase popcorn, and of those who purchase popcorn 20% also purchase a drink.

(a) What is the probability that a randomly selected moviegoer will buy neither popcorn nor a drink?

(b) What proportion of moviegoers purchase a drink (not necessarily *just* a drink)?

Problem 18: A Simple P.M.F.

Suppose X is a random variable with p.m.f. given by

$\boldsymbol{k}$	-2.5	0.3	1.7
$\boldsymbol{p_X(k)}$	0.15	0.25	0.6

(a) Compute $\mathbb{P}(X \geq 0)$

(b) Compute $\mathbb{P}(|X| \leq 2)$

(c) Compute $\mathbb{E}[X]$

(d) Compute $\mathbb{E}\left[\frac{1}{X}\right]$

(e) Compute $\mathrm{Var}(X)$

(f) Compute $\mathrm{Var}(|X|)$

(g) Find $F_X(x)$, the cumulative distribution function (c.d.f.) of X.