PSTAT 120A, Summer 2022: Practice Problems 6

Week 4

Conceptual Review

(a) Geometrically, what does a double integral represent?

(b) How is the evaluation of a double integral contingent on the order of integration?

(c) What is a random vector?

(d) What is a joint p.d.f. / joint p.m.f.?

Problem 1: Picking Points

A point is picked uniformly from the region $\mathcal{R}$ which is the area underneath the portion of the graph of $y = 1 - |x - 1|$ lying in the first quadrant. Let X denote the x−coordinate of this point and let Y denote the y−coordinate of this point.

(a) Find $f_{X,Y}(x, y)$, the joint p.d.f. of (X, Y).

(b) Compute $\mathbb{E}[XY]$.

(c) Find $f_X(x)$ and $f_Y(y)$, the marginal densities of X and Y respectively.

Problem 2: Dice Dice Baby

Suppose Xavier and Yolanda each roll a fair k−sided die (where $k \in \mathbb{N}$). Let X denote the result of Xavier's roll and Y denote the result of Yolanda's roll, and set $Z := \max\{X, Y\}$.

a) Find the p.m.f. of Z.

Hint: First find the c.d.f. of Z and then consider how $F_Z(z)$ and $F_Z(z-1)$ can help us extract the value of $\mathbb{P}(Z = z)$.

b) Verify that your expression from part (a) sums to 1, when summed over the appropriate values. It may be useful to recall that

$$\sum_{i=1}^{w} i = \frac{w(w+1)}{2}$$

c) Compute $\mathbb{E}(Z)$. It may be useful to recall that

$$\sum_{i=1}^{w} i^2 = \frac{w(w+1)(2w+1)}{6}$$

Extra Problems

Problem 3: Continuous Joint Density

Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(x, y) = \begin{cases} c(x + y) & 0 < x < 1, \quad 1 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant.

(a) What is the value of c?

(b) Set up the integral to find $P(X + Y < 3)$. Do not evaluate.

(c) Use the definition of independence to prove whether or not X and Y are independent.

Problem 4: Discrete Joint Density

Suppose that X and Y are jointly distributed discrete random variables with joint probability mass function (P.M.F.) given by

$$p_{X,Y}(x, y) = \begin{cases} \binom{5}{x} \left(\frac{1}{2}\right)^{y+5} & \text{if } x \in \{0, 1, 2, 3, 4, 5\}, \ y \in \{1, 2, \cdots\} \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that $p_{X,Y}(x, y)$ is a valid joint pmf.

(b) Identify the marginal distributions of X and Y **by name**, being sure to include any/all relevant parameter(s).

(c) Use your answer to part (b) to compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(d) Find $\text{Corr}(X, Y)$. **Hint:** based in your answer to part (b), you should not need to do any algebra.

Parts (d) and (e) involve material from Thursday's lecture.

(e) Compute $\mathbb{E}[XY]$. **Hint:** You can either perform a double summation, or you can use your answer to part (d).