PSTAT 120A, Summer 2022: Practice Problems 10: Final Review, Part I

Week 2

Conceptual Review

(a) Review the conceptual questions from the previous Discussion Worksheets!

(b) What are the different notions of conditional p.m.f.'s?

(c) What is the difference between $\mathbb{E}[X \mid Y = y]$ and $\mathbb{E}[X \mid Y]$?

(d) What is the Law of Iterated Expectation? What is the analog for variances?

1 Conditional Distributions and Expectations

Problem 1: Continuous Conditioning

Let (X, Y) be a continuous bivariate random vector with joint p.d.f. given by

$$f_{X,Y}(x, y) = \begin{cases} c \cdot xy & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant.

(a) Find the value of c.

(b) Find $f_Y(y)$, the marginal p.d.f. of Y.

(c) Find $f_{X|Y}(x \mid y)$, the conditional density of X given $Y = y$.

(d) Compute $\mathbb{E}[X]$ using the Law of Iterated Expectations.

(e) Find $f_X(x)$, and verify your answer to part (d).

Problem 2: Discrete Conditioning

Let (X, Y) be a discrete bivariate random vector with joint p.m.f. given by

$$f_{X,Y}(x, y) = \begin{cases} c \cdot xy & \text{if } x \in \{1, 2, 3\},\ y \in \{1, 2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant.

(a) Find the value of c.

(b) Find $p_Y(y)$, the marginal p.m.f. of Y.

(c) Find $p_{X|Y}(x \mid y)$, the conditional p.m.f. of X given $Y = y$.

(d) Compute $p_X(x)$, and determine whether or not X and Y are independent. Try to make an argument using only your answer to part (c), and $p_X(x)$.

Problem 3: Iterations!

In each of the following parts, you will be provided with the conditional distribution of $(X \mid Y)$ and the marginal distribution Y. Using the provided information, compute $\mathbb{E}[X]$ and $\text{Var}(X)$.

(a) $(X \mid Y) \sim \text{Bin}(Y, p); \quad Y \sim \text{Pois}(\mu)$

(b) $(X \mid Y) \sim \text{Exp}(1/Y); \quad Y \sim \text{Gamma}(r, \lambda)$

2 General Problems (includes Cond. Distn's)

Problem 4: Axiomatic Proof

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and events $A, B \in \mathcal{F}$, prove the following identity:

$$\mathbb{P}(A \mid B^{\complement}) = 1 - \frac{\mathbb{P}(A^{\complement})}{\mathbb{P}(B^{\complement})} + \frac{\mathbb{P}(A^{\complement} \cap B)}{\mathbb{P}(B^{\complement})}$$

Problem 5: Variance of Sums

Using only first principles (i.e. taking care **not** to use any previously-derived results pertaining to variance of sums of random variables), derive an expression for

$$\mathrm{Var}\left(\sum_{i=1}^{n} (-1)^i X_i \right)$$

Simplify as much as you can.

Problem 6: Faces of the Same Die

A fair $k-$sided die is rolled n times, where where n and k are fixed natural numbers. Let X denote the number of faces that appear exactly three times. Find $\mathbb{E}[X]$.

Problem 7: A Useful Result

Suppose $X \sim \mathrm{Pois}(\lambda)$ and $Y \sim \mathrm{Pois}(\mu)$ with $X \perp Y$. Find $\mathbb{P}(X = k \mid X + Y = n)$ for natural numbers n and k with $k \leq n$, and use this to recognize the conditional distribution of $(X \mid X + Y = n)$. Be sure to include any/all relevant parameter(s)!

Hint: We know the distribution of $(X + Y)$.

Problem 8: Discrete Joint

Let (X, Y) be a discrete bivariate random vector with joint p.m.f. (probability mass function) given by

$$p_{X,Y}(x, y) = \begin{cases} c \cdot xy & \text{if } x \in \{1, 2, 3\},\ y \in \{1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant.

(a) Find the value of c.

(b) Compute $\mathbb{P}(X = Y)$.

Problem 9: Discrete Convolution

Let $X \sim \text{Geom}(p_1)$ and $Y \sim \text{Geom}(p_2)$ with $X \perp Y$. Derive the p.m.f. of $Z := X+Y$. (**Note:** This will **NOT** be the Negative Binomial p.m.f., unless $p_1 = p_2$)

Problem 10: Waitin' in Line

Alex and Drew are waiting in two separate lines at *Dean* Coffee. Suppose that the time it takes for Alex to reach the counter follows an $\text{Exp}(\lambda_A)$ distribution and the time it takes for Drew to reach the counter $\text{Exp}(\lambda_D)$ distribution. Further suppose that the two lines move independently of each other. What is the probability that Alex reaches the counter before Drew does?

Problem 11: Gamma Gamma Gamma

(a) Show that for any $r > 0$, $\Gamma(r) = (r-1)\Gamma(r-1)$

(b) Use part (a) to argue that $\Gamma(n) = (n-1)!$ whenever $n \in \mathbb{N}$.

(c) Show that $\Gamma(1/2) = \sqrt{\pi}$.

(d) Compute $\Gamma(5/2)$.

Hint: Relate the integral to a Normal density.

Problem 12: It's Giving... Gamma

Prove the following identity:

Hint: Don't try to prove this directly; use probability!

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{\lambda}{\lambda - t}\right)^r \bigg|_{t=0} = \frac{\Gamma(r+n)}{\Gamma(r)\cdot\lambda^n}$$

Problem 13: Sums *(ASV, 9.21)*

Let $X_1, \cdots, X_{500}$ be i.i.d. random variables with expected value 2 and variance 3. The random variables $Y_1, \cdots, Y_{500}$ are independent of the X_i variables, also i.i.d., but they have expected value 2 and variance 2. Use the CLT to estimate

$$\mathbb{P}\left(\sum_{i=1}^{500} X_i > \sum_{i=1}^{500} Y_i + 50\right)$$

Problem 14: Wald's Identity

Prove **Wald's Identity:** if $X_1, X_2, \cdots$ are i.i.d. random variables with finite mean, and N is a nonnegative integer-valued random variable independent of the X_i's (also with finite mean), then

Hint: Condition on $\{N = n\}$

$$\mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbb{E}[N]\cdot\mathbb{E}[X_1]$$

Note that we cannot directly apply linearity, since the upper index of summation is random.

Problem 15: Let's Chalk About It *(modified from ASV 6.3)*

For each lecture, a professor chooses between white, yellow, and purple chalk, independently of previous choices. Each day she chooses white chalk with probability 0.5, yellow chalk with probability 0.4, and purple chalk with probability 0.1.

(a) Suppose we observe this professor for the next 10 days. Define appropriate random variables to count the number of times the professor chooses each color of chalk. Identify **by name** the marginal distributions, taking care to include any/all relevant parameter(s)!

(b) Identify the joint distribution of the random variables you defined in part (a) **by name** (yes, it is a distribution we have encountered before). Be sure to include any/all relevant parameter(s)!

(c) What is the probability that over the next 10 days she will choose white chalk 5 times, yellow chalk 4 times, and purple chalk 1 time?

(d) What is the probability that over the next 10 days she will choose white chalk exactly 9 times?

Problem 16: Continuous Computations

Suppose X is a random variable that has probability density function (p.d.f.) given by

$$f_X(x) = \begin{cases} ce^{-x} & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constnat.

(a) Find the value of c.

(b) Identify the distribution of $Y := X - 2$.

(c) Compute $\mathbb{E}[X]$

(d) Compute $\mathrm{Var}(X)$

(e) Find $F_X(x)$, the c.d.f. of X.

(f) Find the density of $W := X^2$.

(g) Compute $\pi_{0.67}$, the 67$^{\text{th}}$ percentile of the distribution of X.

(h) Find $M_X(t)$, the MGF of X

(i) Let Y be another random variable, independent of X, that has the same p.d.f. as X. Find $f_Z(z)$, the p.d.f. of $Z := X + Y$

(j) Suppose $\{X_i\}_{i=1}^{\infty}$ is an i.i.d. collection of random variables, following the distribution with p.d.f. given by $f_X(x)$ above. If $\bar{X}_{100}$ denotes the sample mean of 100 of these X_i's, approximate $\mathbb{P}(\bar{X}_n > 3)$.

Problem 17: A Useful Result

Let $\{X_i\}$ be an i.i.d. collection of random variables with mean μ and variance σ^2. Show that

$$\mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right] = \sigma^2$$

where $\bar{X}_n := n^{-1}\sum_{i=1}^{n} X_i$ denotes the sample mean.